EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

Hugo Paquet
(LIPN, Paris Nord)

Philip Saville
(Oxford)

these slides available at: philipsaville.co.uk

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

Effectful semantics

EFFECTFUL SEMANTICS

modelling programs

inputs

↓

outputs

1) take in some inputs
2) do some work
3) return some outputs

EFFECTFUL SEMANTICS

modelling programs

inputs

$\downarrow$

outputs

1) take in some inputs
2) do some work
3) return some outputs

eg//

$x : \mathrm{nat} \vdash x + 1 : \mathrm{nat}$

$$\mathbb{N} \longrightarrow \mathbb{N}$$
$$x \longmapsto x + 1$$

EFFECTFUL SEMANTICS

modelling programs

eg//

$x : \mathrm{nat} \vdash x + 1 : \mathrm{nat}$

$$\mathbb{N} \longrightarrow \mathbb{N}$$
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inputs

$\downarrow$

outputs

1) take in some inputs
2) do some work interacting with the world
3) return some outputs

EFFECTFUL SEMANTICS

modelling programs

inputs

↓

outputs

1) take in some inputs
2) do some work interacting with the world
3) return some outputs

effects

- print to screen
- memory
- non-determinism
- probability

...

EFFECTFUL SEMANTICS

modelling programs

inputs → outputs

1) take in some inputs
2) do some work interacting with the world
3) return some outputs

effects

- print to screen
- memory
- non-determinism
- probability

...

eg// $x : nat \vdash \text{print "hi"}; x+1 : nat$

$\rightsquigarrow \mathbb{N} \longrightarrow \mathcal{L}^* \times \mathbb{N} : n \mapsto (hi, n+1)$

EFFECTFUL SEMANTICS

modelling programs

effectful program

inputs (with types)

output type

$$\Gamma \vdash M : A$$

$\rightsquigarrow$ arrow in some structured category

effects

- print to screen
- memory
- non-determinism
- probability

...

PREMONOIDAL & FREYD

PREMONOIDAL & FREYD ~~BI~~CATEGORIES

domains

sets

presheaves

relations

a variety of semantic models

a *unifying framework*

strong monads (Moggi)

premonoidal categories

Freyd categories

(Power-Robinson, Power)

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

2-dimensions = more $\underbrace{\text{refined}}$ semantics

intensional information,
rewrites between programs
(often : via a universal property)

2-dimensions = more $\underbrace{\text{refined}}$ semantics

intensional information,
rewrites between programs

eg//

- maps $A \rightsquigarrow B$ as spans

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

- composition by $\underbrace{\text{pullback}}$

only associative up to iso!

$a \; R \; b$ via x

$$\Leftrightarrow \quad \begin{aligned} f(x) &= a \\ g(x) &= b \end{aligned}$$

2-dimensions = more $\underbrace{\text{refined}}$ semantics

intensional information,
rewrites between programs

effectful program

inputs (with types)

output type

$$\Gamma \vdash M : A$$

$\rightsquigarrow$ 1-cell in some structured bicategory

PREMONOIDAL & FREYD BICATEGORIES

linear logic

rewriting

profunctors

Cat

(co)spans

games

(co)Para

optics, networks, ...

semantics

a variety of semantic models

a unifying framework

~~strong pseudomonads~~

premonoidal bicategories

Freyd bicategories

THIS WORK

The 1-dimensional story

Key point

```
print "a";        ≠        print "b";
  print "b"                  print "a"
```

Key point

$\Gamma \vdash P : A$

$\Delta \vdash Q : B$

run P to v;
run Q to w;
return (v, w)

$\neq$

run Q to w;
run P to v;
return (v, w)

$\Gamma \otimes \Delta \xrightarrow[P \otimes id]{} A \otimes \Delta \xrightarrow[id \otimes Q]{} A \otimes B$

$\Gamma \otimes \Delta \xrightarrow[id \otimes Q]{} \Gamma \otimes B \xrightarrow[P \otimes id]{} A \otimes B$

Key point

$$\Gamma \vdash P : A$$
$$\Delta \vdash Q : B$$

INTERCHANGE FAILS

$$\begin{tikzcd}
 & \Gamma \otimes \Delta \arrow[dl, "P \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes Q"] \arrow[dd, phantom, "\neq"] & \\
A \otimes \Delta \arrow[dr, "\mathrm{id} \otimes Q"'] & & \Gamma \otimes B \arrow[dl, "P \otimes \mathrm{id}"] \\
 & A \otimes B &
\end{tikzcd}$$

run Q,
run P,
pair the results

run P,
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Key point

$$\Gamma \vdash P : A$$
$$\Delta \vdash Q : B$$

INTERCHANGE FAILS

$$\begin{tikzcd}
 & \Gamma \otimes \Delta \arrow[dl, "P \otimes id"'] \arrow[dr, "id \otimes Q"] & \\
A \otimes \Delta \arrow[dr, "id \otimes Q"'] \arrow[rr, phantom, "\neq"] & & \Gamma \otimes B \arrow[dl, "P \otimes id"] \\
 & A \otimes B &
\end{tikzcd}$$

run Q,
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run P,
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interchange holds for pure, effect-free programs

↳ values

a <u>binoidal category</u> $(\mathbb{C}, \otimes)$ has

- $\otimes : \mathrm{ob}\,\mathbb{C} \times \mathrm{ob}\,\mathbb{C} \longrightarrow \mathrm{ob}\,\mathbb{C}$
- $I \in \mathbb{C}$
- for every $A, B \in \mathbb{C}$, functors $A \rtimes (-), (-) \ltimes B : \mathbb{C} \to \mathbb{C}$

  s.t. $A \rtimes B = A \otimes B = A \ltimes B$

a map $f : A \to A'$ is <u>central</u> if

$\forall g.$

$$\begin{tikzcd} & A \otimes B \arrow[dl, "f \otimes B"'] \arrow[dr, "A \otimes g"] & \\ A' \otimes B \arrow[dr, "A' \otimes g"'] & & A \otimes B' \arrow[dl, "f \otimes B'"] \\ & A' \otimes B' & \end{tikzcd}$$

and

$$\begin{tikzcd} & B \otimes A \arrow[dl, "g \otimes A"'] \arrow[dr, "B \otimes f"] & \\ B' \otimes A \arrow[dr, "B' \otimes f"'] & & B \otimes A' \arrow[dl, "g \otimes A'"] \\ & B' \otimes A' & \end{tikzcd}$$

get category $Z(\mathbb{C}) \hookrightarrow \mathbb{C}$

a **premonoidal category** $(\mathbb{C}, \otimes, I)$ has:

binoidal

- $\otimes : \mathrm{ob}\,\mathbb{C} \times \mathrm{ob}\,\mathbb{C} \longrightarrow \mathrm{ob}\,\mathbb{C}$
- $I \in \mathbb{C}$
- for every $A, B \in \mathbb{C}$, functors $A \rtimes (-), (-) \ltimes B : \mathbb{C} \to \mathbb{C}$
  s.t. $A \rtimes B = A \otimes B = A \ltimes B$
- central natural isomorphisms α, λ, ρ
  s.t. $\triangledown$ + ⬡ hold

so $Z(\mathbb{C})$ is monoidal

$(\mathbb{C}, \otimes, I)$ symmetric monoidal

$$t_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$$

+ axioms

eg: for (T, t) strong, $\mathbb{C}_T$ is premonoidal:

$$A \rtimes g := \left(A \otimes B \xrightarrow{A \otimes g} A \otimes TB' \xrightarrow[\text{strength}]{t_{A,B'}} T(A \otimes B') \right)$$

$$f \ltimes B := \left(A \otimes B \xrightarrow{f \otimes B} T(A') \otimes B \longrightarrow T(A' \otimes B) \right)$$

built from t and symmetry

eg: $[\mathbb{C}, \mathbb{C}]_u$ is premonoidal

obj: functors $\mathbb{C} \to \mathbb{C}$

maps: unnatural transformations

ie. families of arrows

$\{\sigma_c : FC \to GC\}_{c \in \mathbb{C}}$

idea

symmetric monoidal $\mathbb{C}$

premonoidal categories axiomatise $\mathbb{C}_T$ ← strong T

= monoidal categories without interchange

idea

symmetric monoidal $\mathbb{C}$

premonoidal categories axiomatise $\mathbb{C}_T$ ⟵ strong T

= monoidal categories without interchange

Freyd categories axiomatise $\mathbb{C} \xrightarrow[\eta \circ -]{} \mathbb{C}_T$

= premonoidal categories with a choice of "centre" a Freyd category has

- $(\mathbb{C}, \otimes, I)$ premonoidal
- $(\mathcal{V}, \otimes, I)$ monoidal (often cartesian)
- $J : \mathcal{V} \longrightarrow \mathbb{C}$ identity on objects,

  s.t.

  ① J lands in $\mathcal{Z}(\mathbb{C})$

  ② J preserves premonoidal structure (strictly) eg:

symmetric monoidal category $\mathbb{C}$

$$\mathbb{C} \xrightarrow[1 \circ -]{} \mathbb{C}_T \quad \text{is a Freyd category}$$

strong monad T

The 2-dimensional story.

a <u>bicategory</u> $\mathcal{B}$ has

- objects $A, B, \ldots$
- 1-cells $f, g, \ldots : A \to B$
- 2-cells $\sigma, \tau : f \Rightarrow g$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd}$$

- natural isomorphisms

$$\left.\begin{aligned} \alpha &: (f \circ g) \circ h \overset{\cong}{\Rightarrow} f \circ (g \circ h) \\ \lambda &: \mathrm{Id} \circ f \overset{\cong}{\Rightarrow} f \\ \rho &: f \circ \mathrm{Id} \overset{\cong}{\Rightarrow} f \end{aligned}\right\}$$ subject to $\triangledown$ + $\hexagon$ laws

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "\mathrm{Id}"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\cong"] & & B \end{tikzcd}$$

"bicategorify"

categories $\rightsquigarrow$ bicategories

equations of maps $\longmapsto$ isomorphisms between maps, subject to equations

hard part: what equations??

examples of bicategories

- Cat (any 2-category)
- Prof, generalised species, ... (any Kleisli bicategory)
- Span($\mathbb{C}$), Rel, ...
- Para($\mathbb{C}$)
- bimodules over a ring

⋮

(Katsumata, Smirnov, Melliès, ...)

eg :

a graded monad is a lax monoidal functor $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]$

grades

$\mu : T_e \circ T_{e'} \Rightarrow T_{e \bullet e'}$

$\eta : id \Rightarrow T_i$

for $(\mathbb{E}, \bullet, i)$ monoidal

eg//

$\mathbb{E} := (\mathbb{N}_{\leq}, 1, \bullet)$

$\mathbb{C} := Set$

$T_n :=$ (lists of length $\leq n$)

(cf. Katsumata)

__eg__ :

a ^strong __graded monad__ is a lax monoidal functor $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]_{\text{strong}}$

grades

$\mu : T_e \circ T_{e'} \Rightarrow T_{e \cdot e'}$
$\eta : id \Rightarrow T_i$

$\Big\}$ strong

eg:

a strong graded monad is a lax monoidal functor $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]_{\text{strong}}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[drr, "g \circ f"'] & T_e B \arrow[r, "T_e g"] & T_e T_{e'} C \arrow[d, "\mu"] \\ & & T_{e \cdot e'} C \end{tikzcd}$$

a version of coPara

get a bicategory $\underline{Kl_T}$:

- obj = those of $\mathbb{C}$
- 1-cells $A \nrightarrow B$ = $(e, A \xrightarrow{f} T_e B)$
- 2-cells $\sigma : f \Rightarrow g$ = regradings $\rightsquigarrow$

$$\begin{tikzcd} & T_e B \arrow[d, "T_\sigma B"] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & T_{e'} B \end{tikzcd}$$

PREMONOIDAL AND FREYD BICATEGORIES

Some bicategorical words

$F(\mathrm{Id}) \cong \mathrm{Id}$

$F(f) \circ F(g) \cong F(f \circ g)$

pseudofunctor = map of bicategories

pseudo natural transformation = map of pseudofunctors

naturality up to isomorphisms

Modification = map of pseudonat. trans.

families of 2-cells + axiom

a **binoidal category** $(\mathbb{C}, \otimes)$ has

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- $I \in \mathbb{C}$
- for every $A, B \in \mathbb{C}$, functors $A \rtimes (-), (-) \ltimes B : \mathbb{C} \to \mathbb{C}$

  s.t. $A \rtimes B = A \otimes B = A \ltimes B$

a <u>binoidal bicategory</u> $(\mathcal{B}, \otimes)$ has

- $\otimes : \mathrm{ob}\,\mathcal{B} \times \mathrm{ob}\,\mathcal{B} \longrightarrow \mathrm{ob}\,\mathcal{B}$
- $I \in \mathcal{B}$
- for every $A, B \in \mathcal{B}$, pseudo functors $A \rtimes (-), (-) \ltimes B : \mathcal{B} \to \mathcal{B}$

  s.t. $A \rtimes B = A \otimes B = A \ltimes B$

a map $f : A \to A'$ is <u>central</u> if

$$\forall g. \quad \begin{tikzcd} & A \otimes B \arrow[dl, "f \ltimes B"'] \arrow[dr, "A \rtimes g"] & \\ A' \otimes B \arrow[dr, "A' \rtimes g"'] & & A \otimes B' \arrow[dl, "f \ltimes B'"] \\ & A' \otimes B' & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \otimes A \arrow[dl, "g \ltimes A"'] \arrow[dr, "B \rtimes f"] & \\ B' \otimes A \arrow[dr, "B' \rtimes f"'] & & B \otimes A' \arrow[dl, "g \ltimes A'"] \\ & B' \otimes A' & \end{tikzcd}$$

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 & A' \otimes B' &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & B \otimes A \arrow[dl, "g \ltimes A"'] \arrow[dr, "B \rtimes f"] & \\
B' \otimes A \arrow[dr, "B' \rtimes f"'] & & B \otimes A' \arrow[dl, "g \ltimes A'"] \\
 & B' \otimes A' &
\end{tikzcd}$$

natural transformations

$A \rtimes (-) \Rightarrow A' \rtimes (-)$ $\qquad$ $(-) \ltimes A \Rightarrow (-) \ltimes A'$

a **central 1-cell** (f, lc^f, rc^f) has

$$\forall g. \quad \begin{tikzcd} & A\otimes B \arrow[dl, "f\ltimes B"'] \arrow[dr, "A\rtimes g"] & \\ A'\otimes B \arrow[dr, "A'\rtimes g"'] \arrow[rr, phantom, "\cong\, lc^f_g"] & & A\otimes B' \arrow[dl, "f\ltimes B'"] \\ & A'\otimes B' & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B\otimes A \arrow[dl, "g\ltimes A"'] \arrow[dr, "B\rtimes f"] & \\ B'\otimes A \arrow[dr, "B'\rtimes f"'] \arrow[rr, phantom, "\cong\, rc^f_g"] & & B\otimes A' \arrow[dl, "g\ltimes A'"] \\ & B'\otimes A' & \end{tikzcd}$$

$A\rtimes(-) \Rightarrow A'\rtimes(-)$ pseudonatural transformations $(-)\ltimes A \Rightarrow (-)\ltimes A'$

a **premonoidal bicategory** $(\mathcal{B}, \otimes, I)$ has:

binoidal

- $\otimes : \mathrm{ob}\,\mathcal{B} \times \mathrm{ob}\,\mathcal{B} \longrightarrow \mathrm{ob}\,\mathcal{B}$
- $I \in \mathcal{B}$
- for every $A, B \in \mathcal{B}$, pseudo functors $A \rtimes (-), (-) \ltimes B : \mathcal{B} \to \mathcal{B}$
  s.t. $A \rtimes B = A \otimes B = A \ltimes B$
- central pseudo natural equivalences α, λ, ρ
- isomorphisms witnessing $\triangledown$, ⬡, ...
  subject to equations

eg//

- $[\mathcal{B},\mathcal{B}]_u$ is premonoidal

  └ pseudofunctors, unnatural transformations, families of 2-cells

- Kl_T is premonoidal

  └ for $T : \mathbb{E} \longrightarrow [\mathbb{C},\mathbb{C}]_{strong}$ a strong graded monad a **Freyd bicategory** has

- $(\mathcal{B}, \otimes, I)$ premonoidal
- $(\mathcal{V}, \otimes, I)$ monoidal
- $J : \mathcal{V} \longrightarrow \mathcal{B}$ identity on objects,

• plus compatibility equations

s.t.

1. J factors through $\mathcal{Z}(\mathcal{B})$
2. J preserves premonoidal structure

up to an icon

how strict? Use examples!

eg//

- for $\mathbb{E} \xrightarrow{T} [\mathbb{C},\mathbb{C}]_{\text{strong}}$ a graded monad

discrete monoidal 2-category

$\mathbb{C} \longrightarrow Kl_T$ is a Freyd bicategory

strong pseudomonad

- [BONUS] $\mathcal{B} \longrightarrow \mathcal{B}_T$ is a Freyd bicategory

Why believe the definition? (Levy)

$$\left(\begin{array}{c}\text{Freyd category} \\ \mathcal{V} \xrightarrow[J]{} \mathbb{C}\end{array}\right) \simeq \left(\begin{array}{l}\text{actions } \triangleright : \mathcal{V} \times \mathbb{C} \to \mathbb{C} \\ \qquad\quad\ \triangleleft : \mathbb{C} \times \mathcal{V} \to \mathbb{C} \\ \text{s.t. } J \text{ is a strict map} \\ \text{of actions:}\end{array}\right)$$

$$\begin{tikzcd}
\mathcal{V} \times \mathbb{C} \arrow[r, "\triangleleft"] & \mathbb{C} & \mathbb{C} \times \mathcal{V} \arrow[l, "\triangleright"'] \\
\mathcal{V} \times \mathcal{V} \arrow[u, "\mathcal{V} \times J"] \arrow[r, "\otimes"'] & \mathcal{V} \arrow[u, "J"'] & \mathcal{V} \times \mathcal{V} \arrow[l, "\otimes"] \arrow[u, "J \times \mathcal{V}"']
\end{tikzcd}$$

Why believe the definition?

A THEOREM

$$\left(\text{Freyd bicategory } \mathcal{V} \xrightarrow[J]{} \mathbb{C} \right) \simeq$$

actions $\triangleright : \mathcal{V} \times \mathbb{C} \to \mathbb{C}$

$\triangleleft : \mathbb{C} \times \mathcal{V} \to \mathbb{C}$

s.t. J is a strict map of actions:

$$\begin{tikzcd}
\mathcal{V} \times \mathbb{C} \arrow[r, "\triangleleft"] & \mathbb{C} & \mathbb{C} \times \mathcal{V} \arrow[l, "\triangleright"'] \\
\mathcal{V} \times \mathcal{V} \arrow[u, "\mathcal{V} \times J"] \arrow[r, "\otimes"'] \arrow[ur, phantom, "\simeq"] & \mathcal{V} \arrow[u, "J"] \arrow[ur, phantom, "\simeq"] & \mathcal{V} \times \mathcal{V} \arrow[l, "\otimes"] \arrow[u, "J \times \mathcal{V}"']
\end{tikzcd}$$

strict = commutes with all the data

all very canonical

philip.saville@cs.ox.ac.uk

SUMMARY

- 2-dimensional models refine 1-dimensional ones
- bicategorical premonoidal + Freyd structure is (part of) a framework for these
- definitions backed by examples and lifting of 1-dimensional correspondence

philip.saville@cs.ox.ac.uk

SUMMARY

- 2-dimensional models refine 1-dimensional ones
- bicategorical premonoidal + Freyd structure is (part of) a framework for these
- definitions backed by examples and lifting of 1-dimensional correspondence

FUTURE WORK: closure, internal language, relation to strengths, coherence, relation to Gray structure, centres, ...