

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

Hugo Paquet
(LIPN, Paris Nord)

Philip Saville
(Oxford)

these slides available at : philipsaville.co.uk .

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

EFFECTFUL SEMANTICS

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

1) take in some inputs

2) do some work

3) return some outputs

EFFECTFUL SEMANTICS

modelling programs

inputs
↓
outputs

- 1) take in some inputs
- 2) do some work
- 3) return some outputs

eg //

$x : \text{nat} \vdash x + 1 : \text{nat}$

↪

$\mathbb{N} \longrightarrow \mathbb{N}$
 $x \mapsto x + 1$

EFFECTFUL SEMANTICS

modelling programs

inputs
↓
outputs

- 1) take in some inputs
- 2) do some work interacting with the world
- 3) return some outputs

eg //

$x : \text{nat} \vdash x + 1 : \text{nat}$

↪

$\mathbb{N} \longrightarrow \mathbb{N}$
 $x \mapsto x + 1$

EFFECTFUL SEMANTICS

modelling programs



inputs
↓
outputs

- 1) take in some inputs
- 2) do some work interacting with the world
- 3) return some outputs

effects

- print to screen
- memory
- non-determinism
- probability
-

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

1) take in some inputs

2) do some work interacting with the world

3) return some outputs

eg // $x : \text{nat} \vdash \text{print } "hi"; x + 1 : \text{nat}$

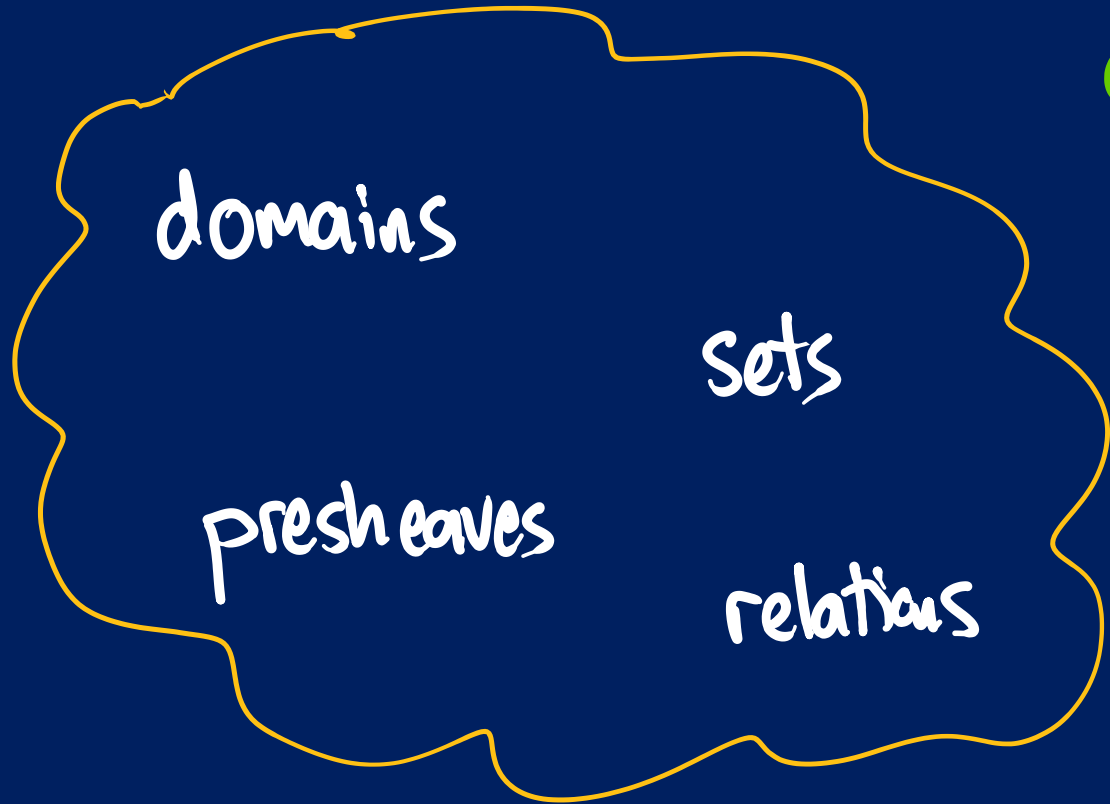
$\rightsquigarrow \mathbb{N} \longrightarrow \mathcal{L}^* \times \mathbb{N} : n \mapsto (hi, n+1)$

effects

- print to screen
- memory
- non-determinism
- probability
-

PREMONOIDAL & FREYD

PREMONOIDAL & FREYD ~~BIC~~CATEGORIES



a variety of
semantic models

↳ a unifying framework

strong monads (Moggi)

premonoidal categories

Freyd categories

(Power-
Robinson,
Power)

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

2-dimensions = more refined semantics

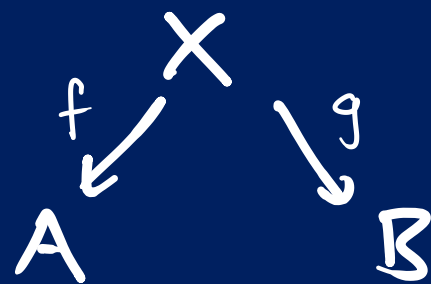
↳ intensional information,
rewrites between programs
(often: via a universal property)

2-dimensions = more refined Semantics

↳ intensional information,
rewrites between programs

eg //

- Maps $A \rightsquigarrow B$ as Spans
- composition by pullback

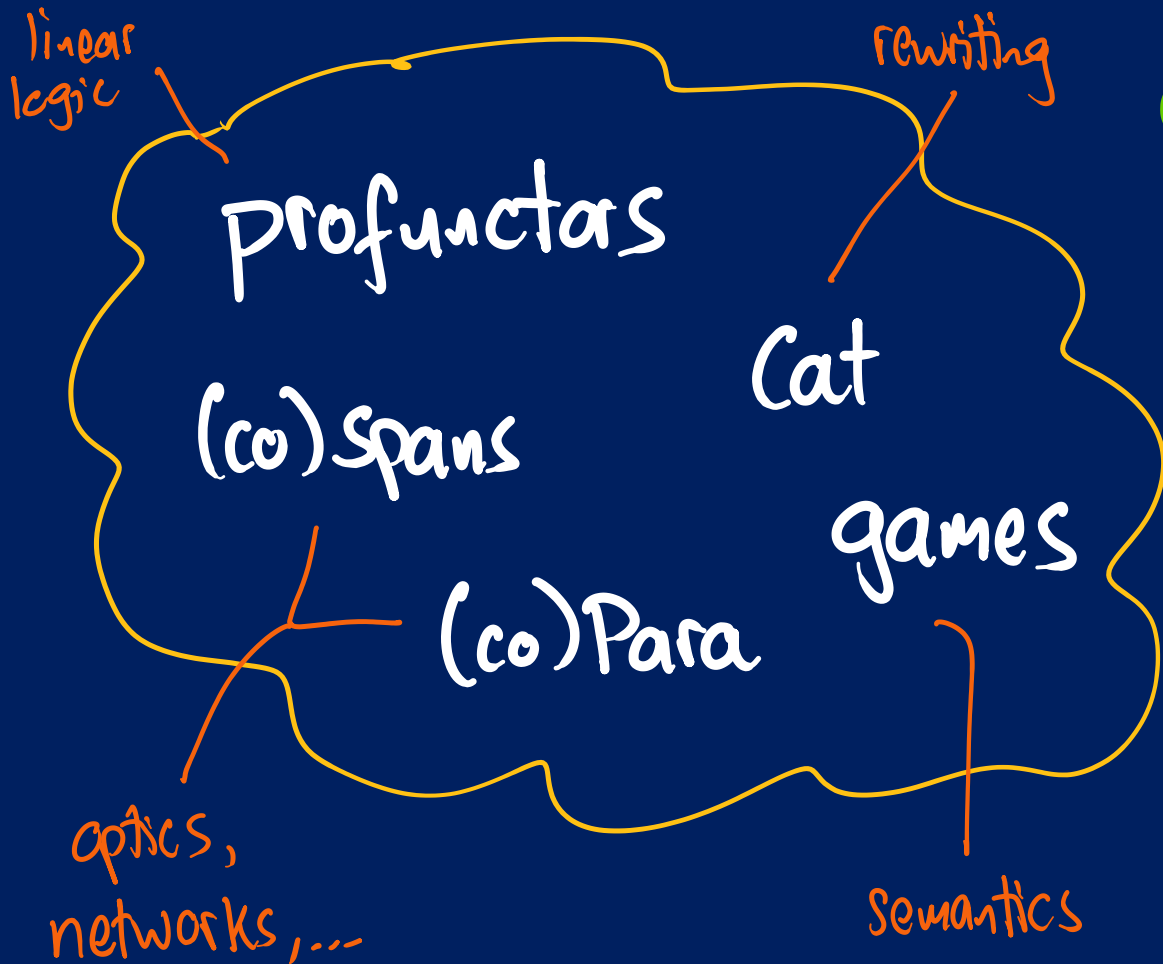


$a R b$ via x

$$\Leftrightarrow \begin{cases} f(x) = a \\ g(x) = b \end{cases}$$

only associative up to Iso!

PREMONOIDAL & FREYD BICATEGORIES



a variety of semantic models



a unifying framework

~~strong pseudomonads~~

premonoidal bicategories

Freyd bicategories

THIS
WORK

Key point

```
print "a";  
print "b"
```

≠

```
print "b";  
print "a"
```

Key point

$$\Gamma \vdash P : A$$

$$\Delta \vdash Q : B$$

run P to V;

run Q to W;

return (v, w)

run Q to W;

run P to V;

return (v, w)

\neq

$$\Gamma \otimes \Delta \xrightarrow{P \otimes \text{id}} A \otimes \Delta \xrightarrow{\text{id} \otimes Q} A \otimes B$$

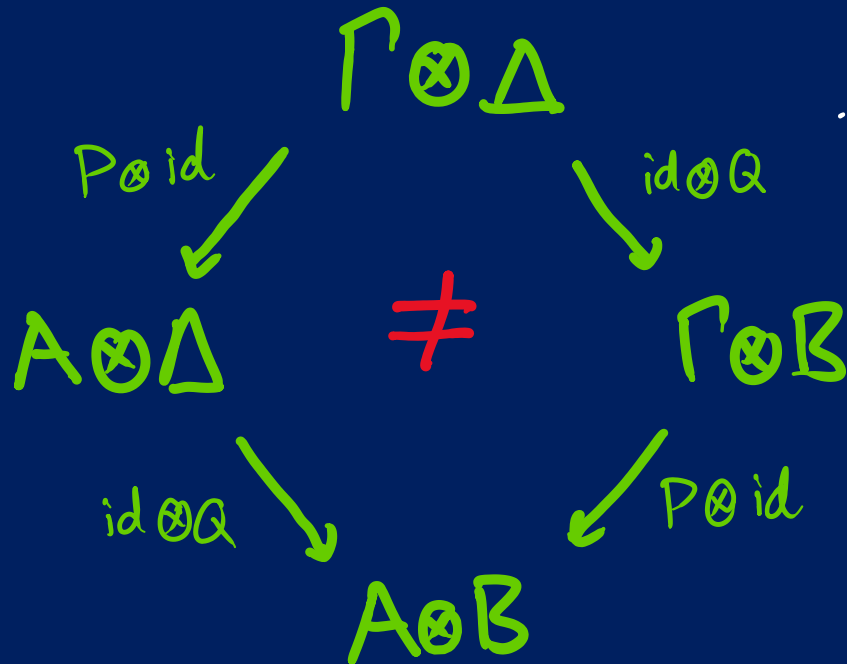
$$\Gamma \otimes \Delta \xrightarrow{\text{id} \otimes Q} \Gamma \otimes B \xrightarrow{P \otimes \text{id}} A \otimes B$$

Key point

$$\Gamma \vdash P : A$$

$$\Delta \vdash Q : B$$

INTERCHANGE FAILS



run Q,
run P,
pair the results

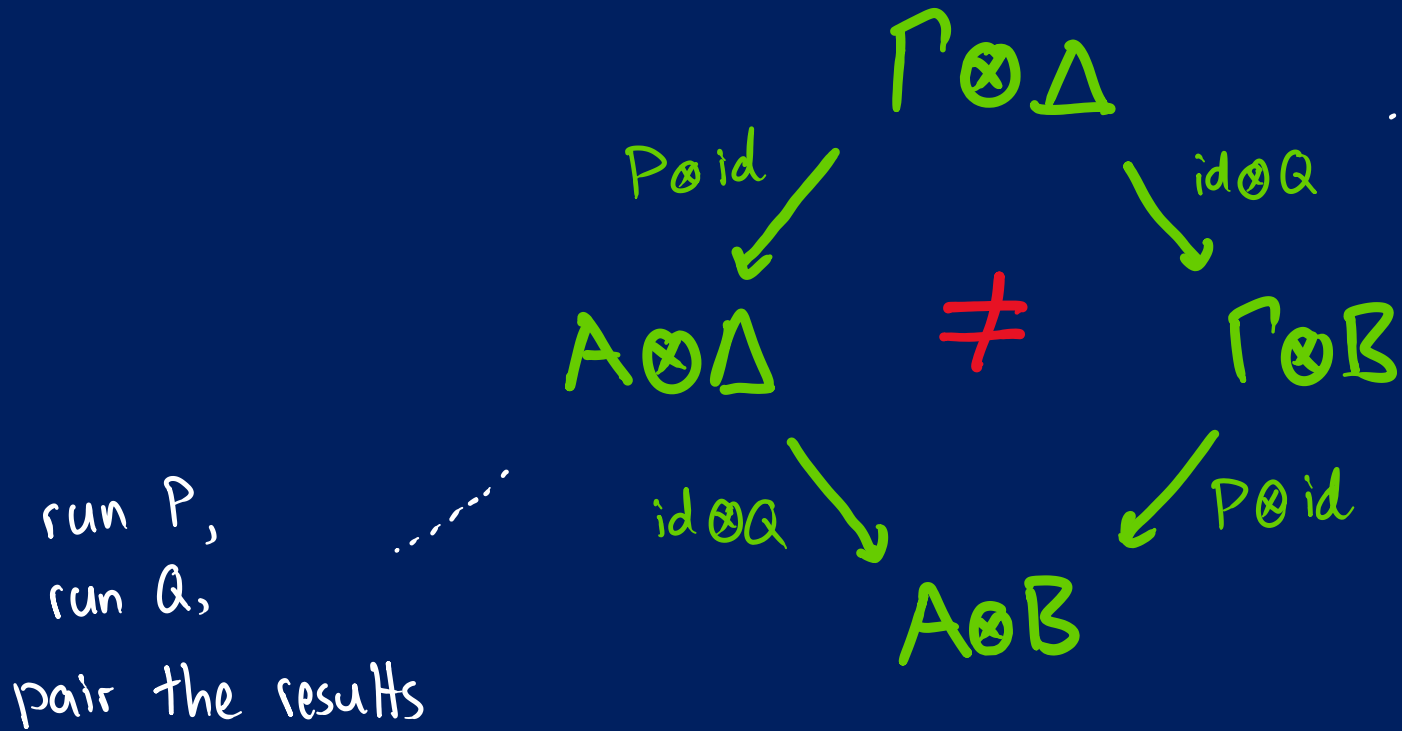
run P,
run Q,
pair the results

Key point

$\Gamma \vdash P : A$

$\Delta \vdash Q : B$

INTERCHANGE FAILS



run Q,
run P,
pair the results

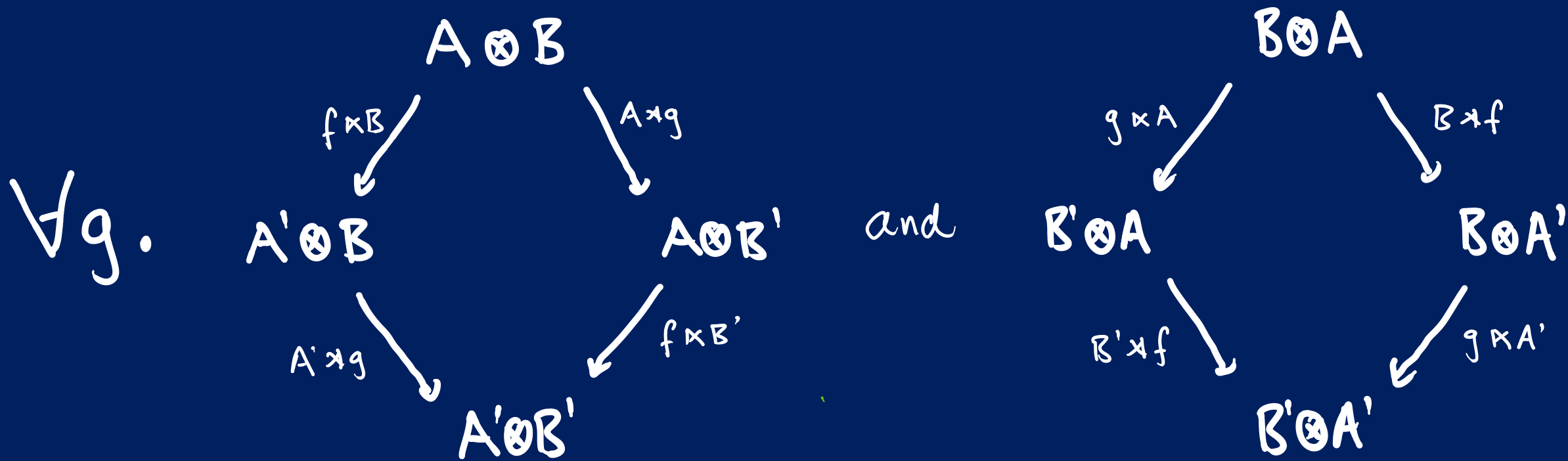
interchange holds
for pure, effect-free
programs

L values

a binoidal category (\mathcal{C}, \otimes) has

- $\otimes : \text{ob } \mathcal{C} \times \text{ob } \mathcal{C} \longrightarrow \text{ob } \mathcal{C}$
- $I \in \mathcal{C}$
- for every $A, B \in \mathcal{C}$, functors $A \rtimes (-), (-) \rtimes B : \mathcal{C} \rightarrow \mathcal{C}$
s.t. $A \rtimes B = A \otimes B = A \times B$

a map $f: A \rightarrow A'$ is central if



get category $Z(\mathcal{C}) \hookrightarrow \mathcal{C}$

a premonoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ has:

binoidal

- $\otimes : \text{ob } \mathcal{C} \times \text{ob } \mathcal{C} \longrightarrow \text{ob } \mathcal{C}$
- $\mathbb{I} \in \mathcal{C}$
- for every $A, B \in \mathcal{C}$, functors $A \rtimes (-), (-) \rtimes B : \mathcal{C} \rightarrow \mathcal{C}$
s.t. $A \rtimes B = A \otimes B = A \times B$
- central natural isomorphisms α, λ, ρ
s.t. $\nabla + \text{hexagon}$ hold

} so $\mathcal{Z}(\mathcal{C})$ is monoidal

(\mathbb{C}, \otimes, I) symmetric monoidal

$$\epsilon_{A,B} : A \otimes B \rightarrow T(A \otimes B)$$

} + axioms

eg: for (T, t) strong, \mathbb{C}_T is premonoidal:

$$A * g := \left(A \otimes B \xrightarrow{A \otimes g} A \otimes TB' \xrightarrow[\text{strength}]{\epsilon_{A, B'}} T(A \otimes B') \right)$$

$$f * B := \left(A \otimes B \xrightarrow{f \otimes B} T(A') \otimes B \xrightarrow{\quad} T(A' \otimes B) \right)$$

} built from t
and symmetry

eg: $[\mathcal{C}, \mathcal{C}]_u$ is premonoidal

$\underbrace{\quad}_{\text{obj}}$: functors $\mathcal{C} \rightarrow \mathcal{C}$

$\underbrace{\quad}_{\text{maps}}$: unnatural transformations

ie. families of arrows


$$\{\sigma_c : FC \rightarrow GC\}_{c \in \mathcal{C}}$$

idea

premonoidal categories
= monoidal categories without interchange

symmetric
monoidal \mathcal{C}

axiomatise \mathcal{C}_T ← strong T



idea

symmetric
monoidal \mathcal{C}

premonoidal categories axiomatise \mathcal{C}_T ← strong
= monoidal categories without interchange

Freyd categories axiomatise $\mathcal{C} \xrightarrow{\eta_0^-} \mathcal{C}_T$
= premonoidal categories with a choice of "centre"

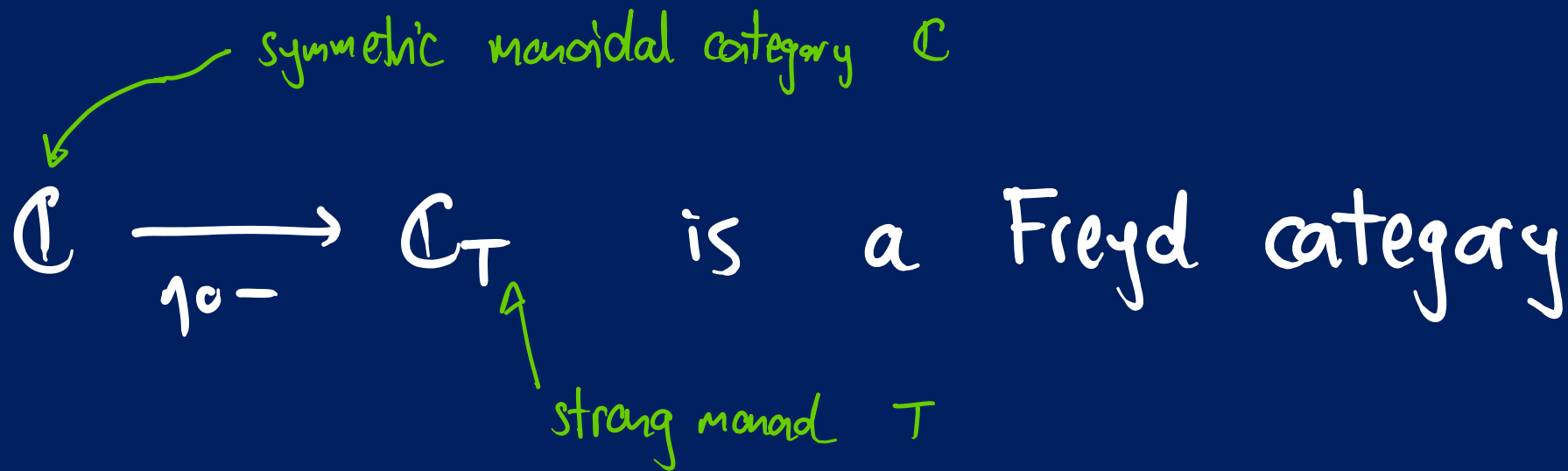
a Freyd category has

- $(\mathcal{C}, \otimes, I)$ premonoidal
- $(\mathcal{V}, \otimes, I)$ monoidal (often cartesian)
- $J: \mathcal{V} \rightarrow \mathcal{C}$ identity on objects,
st.

① J lands in $\mathcal{Z}(\mathcal{C})$

② J preserves premonoidal structure
(strictly)

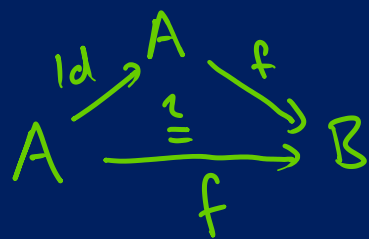
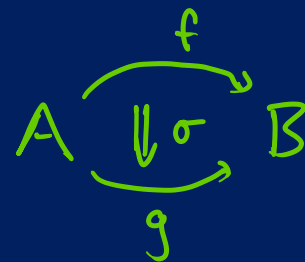
eg :



The 2-dimensional story.

a bicategory \mathcal{B} has

- objects A, B, \dots
- 1-cells $f, g, \dots : A \rightarrow B$
- 2-cells $\sigma, \tau : f \Rightarrow g$
- natural isomorphisms



$$\alpha : (f \circ g) \circ h \xRightarrow{\cong} f \circ (g \circ h)$$

$$\lambda : \text{Id} \circ f \xRightarrow{\cong} f$$

$$g : f \circ \text{Id} \xRightarrow{\cong} f$$

} subject to
 $\triangle + \square$ laws

Categories ^{"bicategorify"} \rightsquigarrow bicategories

equations
of maps



isomorphisms
between maps,
subject to equations

hard part: what equations??

examples of bicategories

- Cat (any 2-category)
- Prof , generalised species, ... (any Kleisli bicategory)
- $\text{Span}(\mathcal{C})$, Rel , ...
- $\text{Para}(\mathcal{C})$
- bimodules over a ring
- \vdots

(Katsumata, Smirnov, Melliès, ...)

eg:

a graded monad is a lax monoidal

functor $T : \mathbb{E} \longrightarrow [\mathcal{C}, \mathcal{C}]$

grades

$$\mu : T_e \circ T_{e'} \Rightarrow T_{e \cdot e'}$$

$$\eta : \text{id} \Rightarrow T_i$$

for (\mathbb{E}, \cdot, i) monoidal

eg//

$$\mathbb{E} := (\mathbb{N}_{\leq}, \perp, \cdot)$$

$$\mathcal{C} := \text{Set}$$

$$T_n := (\text{lists of length } \leq n)$$

(cf. Katsumata)

eg: a ^{strong} graded monad is a lax monoidal
functor $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]$ _{strong}
grades

$$\begin{aligned} \mu : T_e \circ T_{e'} &\Rightarrow T_{e \cdot e'} \\ \eta : \text{id} &\Rightarrow T_i \end{aligned} \quad \left. \vphantom{\begin{aligned} \mu : T_e \circ T_{e'} \\ \eta : \text{id} \end{aligned}} \right\} \text{strong}$$

eg:

a ^{strong} graded monad is a lax monoidal
functor $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]^{\text{strong}}$

get a bicategory Kl_T :

$$\begin{array}{ccc}
 A & \xrightarrow{f} T_e B & \xrightarrow{T_e g} T_e T_{e'} C \\
 & \searrow^{g \circ f} & \downarrow \mu \\
 & & T_{e \cdot e'} C
 \end{array}$$

a version
of coPara

• obj = those of \mathbb{C}

• 1-cells $A \rightarrow B = (e, A \xrightarrow{f} T_e B)$

• 2-cells $\sigma : f \Rightarrow g = \text{regradings}$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & T_e B \\
 & \searrow g & \downarrow T_{\sigma} \\
 & & T_{e'} B
 \end{array}$$

PREMONOIDAL AND FREYD BICATEGORIES

Some bicategorical words

$$F(\text{id}) \cong \text{id}$$

$$F(f) \circ F(g) \cong F(f \circ g)$$

pseudofunctor = map of bicategories

pseudo natural transformation = map of pseudofunctors
naturality up to isomorphisms

modification = map of pseudonat. trans.
families of 2-cells + axiom

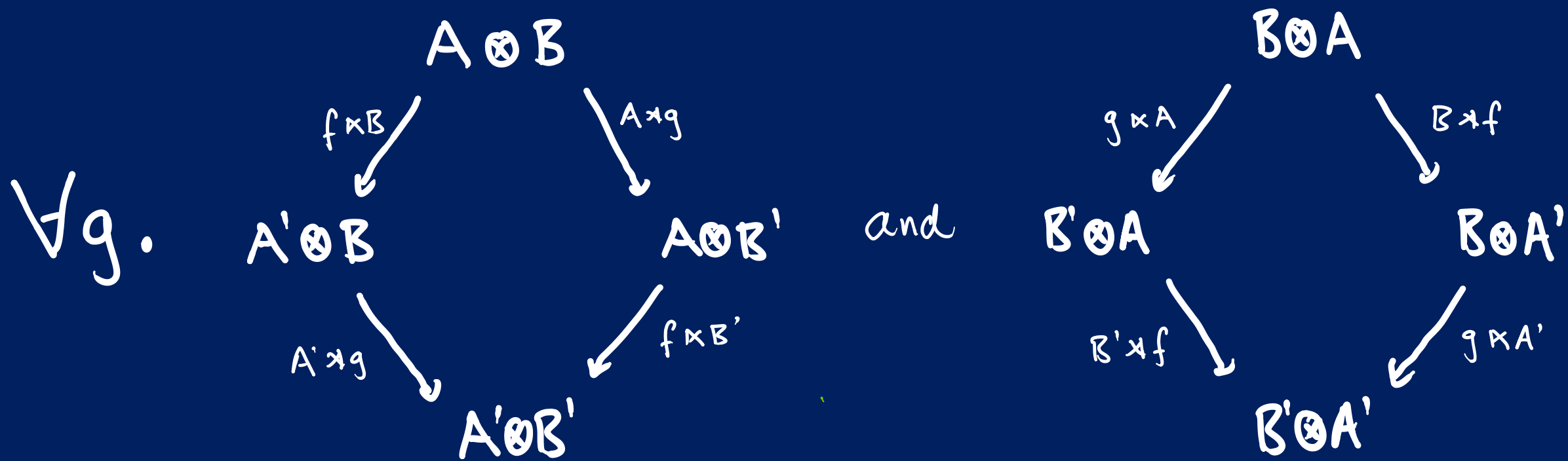
a binoidal category (\mathcal{C}, \otimes) has

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- for every $A, B \in \mathcal{C}$, functors $A \rtimes (-), (-) \rtimes B : \mathcal{C} \rightarrow \mathcal{C}$
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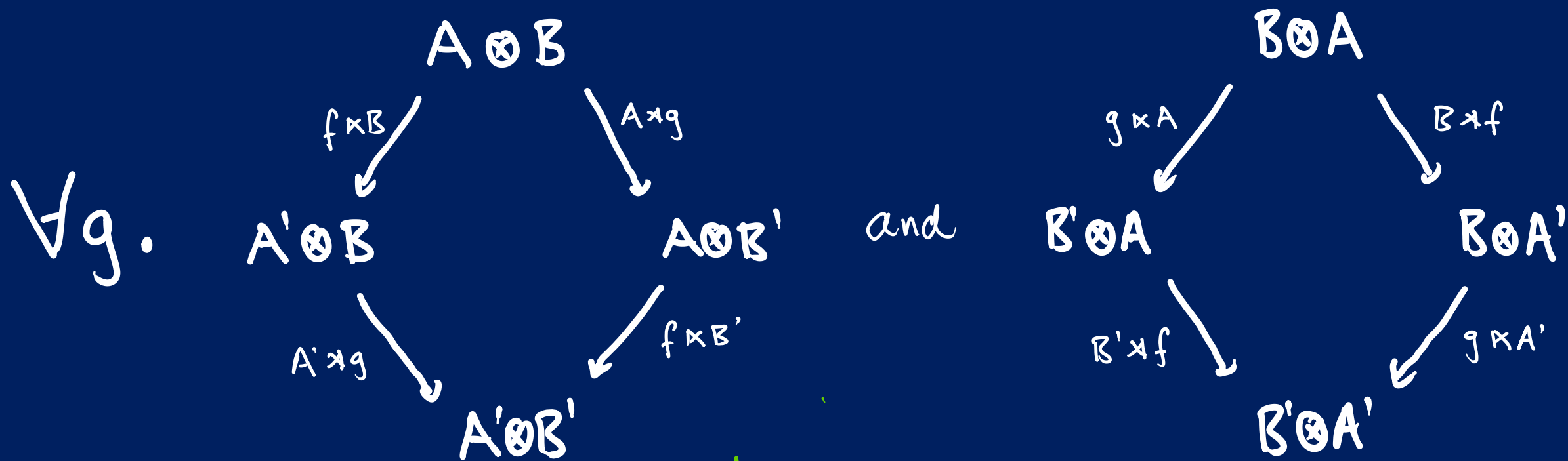
a binoidal bicategory (\mathcal{B}, \otimes) has

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- $I \in \mathcal{B}$
- for every $A, B \in \mathcal{B}$, ^{pseudo} functors $A \rtimes (-), (-) \ltimes B : \mathcal{B} \rightarrow \mathcal{B}$
s.t. $A \rtimes B = A \otimes B = A \ltimes B$

a map $f: A \rightarrow A'$ is central if

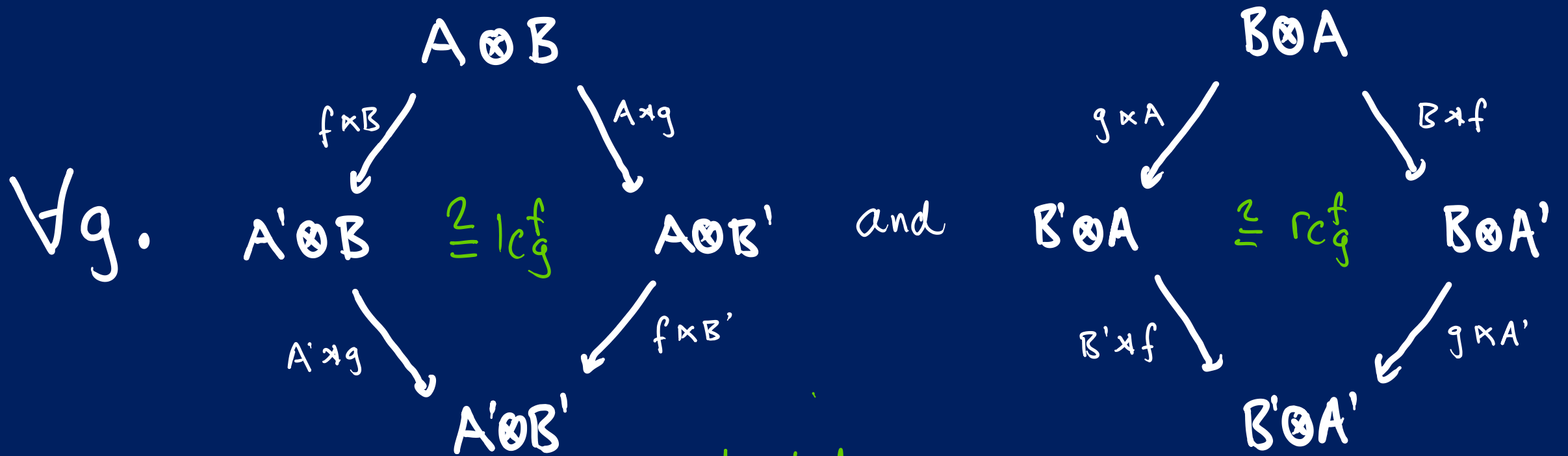


a map $f: A \rightarrow A'$ is central if



$A \times (-) \Rightarrow A' \times (-)$
natural transformations
 $(-) \times A \Rightarrow (-) \times A'$

a central 1-cell (f, lc^f, rc^f) has



$A \times (-) \Rightarrow A' \times (-)$

pseudonatural transformations

$(-) \times A \Rightarrow (-) \times A'$

a premonoidal bicategory $(\mathcal{B}, \otimes, \mathbb{I})$ has:

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s.t. $A \rtimes B = A \otimes B = A \ltimes B$
- central ^{pseudo} natural equivalences α, λ, ρ
- isomorphisms witnessing $\triangleright, \triangleleft, \dots$
subject to equations



eg //

• $[B, B]_u$ is premonoidal

↳ pseudofunctors, unnatural transformations,
families of 2-cells

• Kl_T is premonoidal

↳ for $T: \mathbb{E} \rightarrow [C, C]_{\text{strong}}$
a strong graded monad

a Freyd bicategory has

- $(\mathcal{B}, \otimes, \mathbb{I})$ premonoidal
 - $(\mathcal{V}, \otimes, \mathbb{I})$ monoidal
 - $J: \mathcal{V} \rightarrow \mathcal{B}$ identity on objects,
st.
- plus compatibility equations

① J factors through $\mathcal{Z}(\mathcal{B})$

② J preserves premonoidal structure
up to an icon



how strict? use examples!

eg//

• for $\mathbb{E} \xrightarrow{T} [\mathbb{C}, \mathbb{C}]_{\text{strong}}$ a graded monad

discrete monoidal 2-category $\mathbb{C} \longrightarrow \mathbf{Kl}T$ is a Freyd bicategory

• *[BONUS]* $\mathbb{B} \longrightarrow \mathbb{B}_T$ is a Freyd bicategory *strong pseudomonad*

Why believe the definition?

(Levy)

(Freyd category)
 $\mathcal{V} \xrightarrow{\mathcal{J}} \mathcal{C}$

\cong

actions $\Delta : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$

$\triangleleft : \mathcal{C} \times \mathcal{V} \rightarrow \mathcal{C}$

s.t. \mathcal{J} is a strict map
of actions:

$$\begin{array}{ccccc} \mathcal{V} \times \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} & \xleftarrow{\triangleleft} & \mathcal{C} \times \mathcal{V} \\ \mathcal{V} \times \mathcal{J} \uparrow & & \uparrow \mathcal{J} & & \uparrow \mathcal{J} \times \mathcal{V} \\ \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes} & \mathcal{V} & \xleftarrow{\otimes} & \mathcal{V} \times \mathcal{V} \end{array}$$

Why believe the definition?

A THEOREM

(Freyd **bicategory**) \cong
 $\mathcal{V} \xrightarrow{\mathcal{J}} \mathcal{C}$

all very
canonical

actions $\Delta : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$

$\triangleleft : \mathcal{C} \times \mathcal{V} \rightarrow \mathcal{C}$

s.t. \mathcal{J} is a strict map
of actions:

$$\begin{array}{ccccc} \mathcal{V} \times \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} & \xleftarrow{\triangleleft} & \mathcal{C} \times \mathcal{V} \\ \mathcal{V} \times \mathcal{J} \uparrow & \cong & \uparrow \mathcal{J} & \cong & \uparrow \mathcal{J} \times \mathcal{V} \\ \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes} & \mathcal{V} & \xleftarrow{\otimes} & \mathcal{V} \times \mathcal{V} \end{array}$$

strict = commutes with all the data

SUMMARY

- 2-dimensional models refine 1-dimensional ones
- bicategorical premonoidal + Freyd structure
is (part of) a framework for these
- definitions backed by examples and lifting
of 1-dimensional correspondence

SUMMARY

- 2-dimensional models refine 1-dimensional ones
- bicategorical premonoidal + Freyd structure is (part of) a framework for these
- definitions backed by examples and lifting of 1-dimensional correspondence

FUTURE WORK : closure, internal language, relation to strengths, coherence, relation to Gray structure, centres, ...