Effectful semantics in bicategories: strong, commutative, and concurrent pseudomonads *

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ABSTRACT

We develop the theory of strong and commutative monads in the 2-dimensional setting of bicategories. This provides a framework for the analysis of effects in many recent models which form bicategories and not categories, such as those based on profunctors, spans, or strategies over games.

We then show how the 2-dimensional setting provides new insights into the semantics of concurrent functional programs. We introduce concurrent pseudomonads, which capture the fundamental weak interchange law connecting parallel composition and sequential composition. This notion brings to light an intermediate level, strictly between strength and commutativity, which is invisible in traditional categorical models. We illustrate the concept with the continuation pseudomonad in concurrent game semantics.

In developing this theory, we take care to understand the coherence laws governing the structural 2-cells. We give many examples and prove a number of practical and foundational results.

KEYWORDS

Semantics, effect, monad, strength, concurrency, bicategory

1 INTRODUCTION

Moggi [57, 58] famously observed that the structure of effectful computation is captured by the category-theoretic notion of *strong monad*. This gives a framework for constructing new models and relating existing ones, abstracting away from any particular effect. This paper lays the foundations for modelling effects using monads in *2-dimensional* category theory, where one has not just morphisms between objects, but also morphisms between morphisms (Sections 1.1 and 2). We have two motivations:

(1) Many recent semantic models are not categories but *bicategories* (*e.g.* [6, 15, 16, 54]). However, we lack a unifying, Moggi-style framework for them. The time is right to set up the proper theoretical foundations for these models.
(2) Some well-known effects are already 2-categorical (see Sections 1.1 and 1.3). Making this structure explicit lets us see them as instances of a larger pattern, highlighting new connections, theoretical insights, and examples.

In this paper we lift Moggi's foundational framework to the 2-dimensional setting (Sections 4 and 5), and show this is a suitable setting for modelling effectful programs (Section 5.3). In doing so, we discover new notions that are invisible in 1-dimensional approaches (Section 6). Throughout we give plenty of examples (*e.g.* Sections 4.3 and 6.2) and take care to mathematically justify our choice of definitions (Section 7).

1.1 Semantics in 2-dimensional categories

A 2-dimensional category comes with objects $(A, B, \ldots)$, morphisms $(f, g, \ldots : A \to B)$, often called *1-cells*, and *2-cells* $(\sigma, \tau, \ldots : f \Rightarrow g)$ between the 1-cells. There are various kinds of 2-dimensional categories. In this paper we work with *bicategories*, a general notion in which the associativity and identity laws for the composition of morphisms only hold up to isomorphism.

Bicategories typically arise when the composition of morphisms uses a universal property (*e.g.* a categorical limit or colimit), because it is then determined only up to isomorphism. There are many examples from semantics: game semantics [6, 54], recent models of linear logic based on profunctors [15, 16, 20], and models describing the $\beta\eta$-rewrites of the simply-typed λ-calculus [17, 30, 69]. These models come with more structure, and typically provide finer-grained or more intensional information than categorical ones. (See also Section 2 for detailed examples.)

In addition to these recent models, many traditional categories from semantics are already 2-dimensional:

Domain theory: The basic idea of domain theory is to model recursion using a partial order on sets of continuous functions. This is a simple form of 2-dimensional structure on categories of domains, but there is a rich theory (*e.g.* [32, 72, 77]).

Non-determinism: Perhaps the simplest model for non-determinism is the category of sets and relations, where programs correspond to functions $A \to \mathcal{P}(B)$. The inclusion order on relations gives 2-dimensional structure with a natural semantic interpretation in terms of possible returned values.

Concurrency. Maps of processes play an central role in models of concurrency based on event structures or presheaves [7, 81], and *concurrent Kleene Algebra* is similarly based on a partial order over processes [31].

Note that 2-dimensional aspects are also relevant on the syntactic side (see [17, 38, 59]), and other 2-dimensional notions are also important, *e.g.* lax 2-dimensional functors for comparing models [2, 10].

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1.2 The monadic theory of effects

We recall the traditional framework (*e.g.* [57, 58]). A strong monad on a monoidal category $(\mathbb{C}, \otimes, I)$ is a monad (T, μ, η) equipped with natural transformations

$$A \otimes T(B) \xrightarrow{t_{A,B}} T(A \otimes B) \qquad T(A) \otimes B \xrightarrow{s_{A,B}} T(A \otimes B)$$

called the *left strength* and the *right strength*, compatible with both the monoidal structure of $\mathbb{C}$ and the monad structure of T (see *e.g.* [40, 51]). An effectful program $(\Gamma \vdash M : A)$ is then modelled by a Kleisli arrow $\Gamma \to TA$ in $\mathbb{C}$.

The strength makes substitution possible even in the presence of free variables. For example, we can substitute M for a variable $x : A$ in another program $(\Delta, x : A \vdash N : B)$ using the strength and the Kleisli extension operation:

$$\Delta \otimes \Gamma \xrightarrow{\Delta \otimes M} \Delta \otimes TA \xrightarrow{t_{\Delta,A}} T(\Delta \otimes A) \xrightarrow{\gg= N} TB.$$

This paper is about a notion of pseudostrength for 2-dimensional pseudomonads, where *pseudo* indicates that the equations in the definition of a strong monad have been replaced by isomorphisms. These isomorphisms must in turn satisfy a number of equations, which we justify in various ways; see Section 7.

1.3 Pseudo monoidality and lax monoidality: commutativity and concurrency

The theory of strong monads provides a basis for reasoning about sequential composition. A natural question is whether the order of execution matters for the two components of a pair: if $(\Gamma \vdash M : A)$ and $(\Delta \vdash N : B)$ are effectful programs then typically the program

$$\Gamma, \Delta \vdash (M, N) : A \otimes B$$

behaves differently depending on which component is evaluated first. (We model contexts linearly to remain as general as possible, since categories with products are instances of monoidal categories. But this is orthogonal to the topic of this paper.)

1.3.1 Commutativity. An effect is called *commutative* if the choice of evaluation order for pairs has no impact on program behaviour. Probability and divergence are commutative; printing and state are not. Correspondingly, a monad is called *commutative* when the equation

$$\begin{array}{ccccc} & & TA \otimes TB & & \\ & \swarrow^{s} & & \searrow^{t} & \\ T(A \otimes TB) & & & & T(TA \otimes B) \\ {\scriptstyle Tt}\downarrow & & & & \downarrow{\scriptstyle Ts} \\ TT(A \otimes B) & & & & TT(A \otimes B) \\ & \searrow_{\mu} & & \swarrow_{\mu} & \\ & & T(A \otimes B) & & \end{array} \qquad (1)$$

holds. This is a semantic counterpart to the property that the evaluation order for pairs does not affect program behaviour: commutative monads model commutative effects. In Section 5 we will define commutative pseudomonads by replacing (1) with an invertible 2-cell, subject to coherence axioms.

1.3.2 Monoidality. Kock [39, 40] showed that, for a commutative monad T, the family of maps

$$\chi_{A,B} : TA \otimes TB \longrightarrow T(A \otimes B) \qquad (2)$$

defined by either of the routes around (1) gives T the structure of a *monoidal* monad; and that, conversely, given maps as in (2) satisfying suitable equations we can recover a commutative strength for T. In this paper we prove a general 2-categorical version of Kock's theorem (Theorem 5.9): pseudomonoidality of a pseudomonad corresponds to pseudocommutativity.

1.3.3 Concurrency. By moving to a 2-dimensional setting we can give a presentation of concurrency. The starting observation is that a monoidal structure for T could be used to evaluate program fragments in parallel:

$$P \mid\mid Q \;\; := \;\; \Gamma \otimes \Delta \xrightarrow{P \otimes Q} TA \otimes TB \xrightarrow{\chi} T(A \otimes B)$$

By Kock's theorem, this parallel evaluation is semantically indistinguishable from either of the two sequential executions: modelling concurrency in this way forces the effect to be commutative.

In a 2-dimensional category, however, we can weaken the notion of monoidality to obtain a setting in which programs with *non-commutative* effects can be evaluated in parallel, according to a 2-dimensional constraint:

$$\begin{array}{ccccc} & & TA \otimes TB & & \\ & \swarrow^{s} & & \searrow^{t} & \\ T(A \otimes TB) & & \Big| & & T(TA \otimes B) \\ {\scriptstyle Tt}\downarrow & \Longrightarrow & \chi & \Longleftarrow & \downarrow{\scriptstyle Ts} \\ TT(A \otimes B) & & \downarrow & & TT(A \otimes B) \\ & \searrow_{\mu} & & \swarrow_{\mu} & \\ & & T(A \otimes B) & & \end{array}$$

The 2-cells above are not invertible in general, and do not make the pseudomonad commutative. Replacing the equation (1) by a pair of non-invertible 2-cells, as above, corresponds to replacing the equation $(P \mid\mid Q) \,;\, (P' \mid\mid Q') = (P \,;\, P') \mid\mid (Q \,;\, Q')$ relating sequential and parallel composition of processes by the *weak interchange law* for parallel and sequental composition

$$(P \mid\mid Q) \,;\, (P' \mid\mid Q') \implies (P \,;\, P') \mid\mid (Q \,;\, Q') \qquad (3)$$

attributed to Hoare, Möller, Struth, and Wehrman [31]. This law is a basic feature of maps in models of concurrency. Intuitively, the program on the left has more dependencies—and so may have fewer traces—than the right one: see Figure 1 for an illustration with event structures (made formal in Section 6.2).

The 2-categorical nature of the weak interchange law is already appreciated (see [55]); in this paper we reframe it in the general context of 2-dimensional monad theory and computational effects. We show that the appropriate monadic abstraction for modelling concurrency is a particular class of lax monoidal pseudomonads, in which certain structural 2-cells are not required to be invertible. These are a fully 2-dimensional generalisation of the concurrent monads of Rivas & Jaskelioff [67]. Accordingly, we call these *concurrent pseudomonads* (Definition 6.1).

Our concurrent pseudomonads are always strong (Proposition 6.3) and, as we explain, in the Kleisli bicategory for a concurrent pseudomonad, the premonoidal structure determines a lax functor $\otimes$

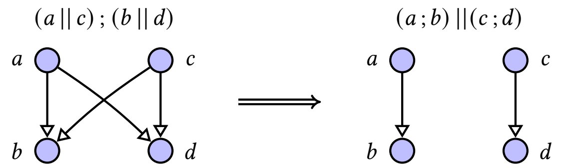


Figure 1: The weak interchange law of sequential and parallel composition, as a map of event structures (see Section 6.2).

of two arguments (Proposition 6.4). This corresponds precisely to requiring a 2-cell as in (3).

1.4 Outline

We begin with an introduction to bicategories and their basic theory (Sections 2 and 3). We then introduce a new definition of strong pseudomonads (Section 4), and illustrate this with plenty of examples (Section 4.3).

We then turn to commutative and monoidal structure (Section 5). We define monoidal pseudomonads and generalise Hyland & Power's definition for commutative pseudomonads [33], then prove a version of Kock's theorem that the two are interchangeable (Theorem 5.9). We also explore the structure of the Kleisli bicategory for strong and commutative pseudomonads (Section 5.3).

In Section 6 we introduce concurrent pseudomonads and show they are strong; we also observe their Kleisli bicategory does indeed model the weak interchange law (3). Section 6.2 illustrates the key ideas with an extended example in concurrent game semantics.

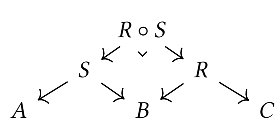
Finally, in Section 7 we put the definitions in their proper mathematical context—namely, as internal pseudomonads—and establish a form of coherence result. Together, these give us confidence in the correctness of our definitions, especially the often-subtle question of how to choose coherence axioms on the 2-cells.

The appendices contain details and proof-sketches omitted from the main body for reasons of space.

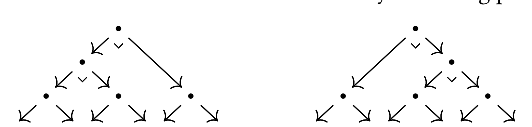
2 TWO EXAMPLES OF BICATEGORIES

As an introduction to bicategories, we consider two illustrative examples. First we look at a model based on spans. Spans occur widely in models of programming languages and computational processes (*e.g.* [1, 14, 23, 54]).

Example. Spans of sets. Consider a model in which objects are sets and a morphism from A to B consists of a set S and a span of functions $A \leftarrow S \rightarrow B$. We can compose a pair of morphisms $A \leftarrow S \rightarrow B$ and $B \leftarrow R \rightarrow C$ using a pullback of functions:



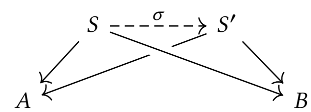
This correctly captures a notion of 'plugging together' spans but is only associative in a weak sense: the two ways of taking pullbacks



(4)

are not generally equal, but they can be shown to be isomorphic by the universal property that defines pullbacks. Similarly, the span $A \xleftarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} A$ is only a weak identity for composition, because pulling back along id only gives an isomorphic set.

To describe the laws of composition in this model, therefore, we require a notion of morphism between spans. If S and S' are spans from A to B, then a map between them is a function $\sigma : S \to S'$ that commutes with the span legs on each side:



The two iterated composites in (4) are isomorphic as spans, so composition of spans is associative up to isomorphism. Similarly, the identity span is unital up to isomorphism. Because these isomorphisms arise from a universal property, they behave well together. Bicategories axiomatise this situation.

Definition 2.1 ([3]). A bicategory $\mathscr{B}$ consists of:

- A collection of objects $A, B, \ldots$
- For all objects A and B, a collection of morphisms from A to B, themselves related by morphisms: thus we have a *hom-category* $\mathscr{B}(A, B)$ whose objects (typically denoted $f, g : A \to B$) are called *1-cells*, and whose morphisms (typically denoted $\sigma, \tau : f \Rightarrow g$) are called *2-cells*. The category structure means we can compose 2-cells between parallel 1-cells.
- For all objects A, B, and C, a composition functor $\circ_{A,B,C}$: $\mathscr{B}(B, C) \times \mathscr{B}(A, B) \longrightarrow \mathscr{B}(A, C)$ and, for all A, an identity 1-cell $\mathsf{Id}_A \in \mathscr{B}(A, A)$.
- Coherent structural 2-cells: since the composition of 1-cells is weak, we have a natural family of invertible 2-cells $\mathsf{a}_{f,g,h}$: $(f \circ g) \circ h \Longrightarrow f \circ (g \circ h)$ instead of the usual associativity equation. Similarly, we have natural families of invertible 2-cells $\mathsf{l}_f : \mathsf{Id}_B \circ f \Longrightarrow f$ and $\mathsf{r}_f : f \circ \mathsf{Id}_A \Longrightarrow f$ instead of the left and right identity laws. These 2-cells must satisfy coherence axioms similar to those for a monoidal category.

To illustrate further we consider the **Para** construction, which is a general way to build models of parametrized processes [19, 29] (see also [4, 11, 12]). In this bicategory, the 2-cells are reparametrizations, and the weakness arises because we are tracking extra information. We will use this bicategory several times, so we spell out the definition in detail.

Example: the **Para** *construction.* Starting from a monoidal category $(\mathbb{C}, \otimes, I)$, the bicategory $\mathbf{Para}(\mathbb{C})$ is defined as follows:

- The objects are those of $\mathbb{C}$.
- A 1-cell from A to B is a parametrized $\mathbb{C}$-morphism, defined as an object $P \in \mathbb{C}$ together with a morphism $f : P \otimes A \to B$ in $\mathbb{C}$. The object P is thought of as a space of parameters.
- A 2-cell from $f : P \otimes A \to B$ to $g : P' \otimes A \to B$ is a reparametrization map, *i.e.* a map $\sigma : P \to P'$ such that $g \circ (\sigma \otimes A) = f$.

Composition of 1-cells is defined using the tensor product of parameters: if $f : P \otimes A \to B$ and $g : Q \otimes B \to C$, then $g \circ f$ is the object $Q \otimes P$ equipped with the map

$$(Q \otimes P) \otimes A \xrightarrow{\cong} Q \otimes (P \otimes A) \xrightarrow{Q \otimes f} Q \otimes B \xrightarrow{g} C$$

where the first map is the associativity of the tensor product.

If we also have $h : R \otimes C \to D$, then the two composites $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have parameter spaces $(R \otimes Q) \otimes P$ and $R \otimes (Q \otimes P)$, respectively. Because the tensor product in a monoidal category is generally associative only up to isomorphism, these 1-cells are only isomorphic in $\mathbf{Para}(\mathbb{C})$. A similar argument applies to the identity laws, so $\mathbf{Para}(\mathbb{C})$ is a bicategory with associativity and unit isomorphisms given by $\mathbb{C}$'s monoidal structure.

3 PSEUDOFUNCTORS, PSEUDOMONADS, AND MONOIDAL BICATEGORIES

Many concepts in category theory have corresponding versions for bicategories. We first summarise the basic definitions of pseudofunctors, pseudonatural transformations, and modifications (Section 3.1), then discuss the bicategorical notions of monad (Section 3.2) and monoidal structure (Section 3.3) needed for this paper. For reasons of space we only give a brief outline and omit the coherence axioms. For a full overview of the basic bicategorical definitions, see [44]; for the definition of (symmetric) monoidal bicategories, including many beautiful diagrams, see [71]. Gentle introductions to the wider subject of bicategories include [3, 35]; a more theoretical-computer science perspective is available in [64, 65].

3.1 Basic notions

Morphisms of bicategories are called pseudofunctors. Just as bicategories are categories 'up to isomorphism', so pseudofunctors are functors 'up to isomorphism'.

Definition 3.1. A *pseudofunctor* $F : \mathscr{B} \to \mathscr{C}$ consists of:

- A mapping $F : ob(\mathscr{B}) \to ob(\mathscr{C})$ on objects;
- A functor $F_{A,B} : \mathscr{B}(A, B) \to \mathscr{C}(FA, FB)$ for each $A, B \in \mathscr{B}$;
- A *unitor* $\psi_A : \mathsf{Id}_{FA} \overset{\cong}{\Longrightarrow} F(\mathsf{Id}_A)$ for each $A \in \mathscr{B}$;
- A *compositor* $\phi_{f,g} : F(f) \circ F(g) \overset{\cong}{\Longrightarrow} F(f \circ g)$ for every composable pair of 1-cells f and g, natural in f and g.

This data is subject to three axioms similar to those for strong monoidal functors (see *e.g.* [44]).

We generally abuse notation by referring to a pseudofunctor (F, ϕ, ψ) simply as F; where there is no risk of confusion, we shall employ similar abuses for structure throughout. A pseudofunctor is called *strict* if ϕ and ψ are both the identity.

Example 3.2. Every endofunctor F on a monoidal category $(\mathbb{C}, \otimes, I)$ with a strength $t_{A,B} : A \otimes F(B) \to F(A \otimes B)$ (see *e.g.* [40]) determines a strict endo-pseudofunctor $\widetilde{F}$ on $\mathbf{Para}(\mathbb{C})$. The action on objects is the same, and on 1-cells $\widetilde{F}(P \otimes A \xrightarrow{f} B)$ is the object P together with the composite $\big(P \otimes FA \xrightarrow{t} F(P \otimes A) \xrightarrow{Ff} FB\big)$.

Transformations between pseudofunctors are like natural transformations, except one must say in what sense naturality holds for each 1-cell.

Definition 3.3. For pseudofunctors $F, G : \mathscr{B} \to \mathscr{C}$, a *pseudonatural transformation* $\eta : F \Rightarrow G$ consists of:

- A 1-cell $\eta_A : FA \to GA$ for every $A \in \mathscr{B}$;
- For every $f : A \to B$ in $\mathscr{B}$ an invertible 2-cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & \overset{\overline{\eta}_f}{\Leftarrow} & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array} \qquad (5)$$

natural in f and satisfying identity and composition laws.

Example 3.4. Every natural transformation $\sigma : F \Rightarrow F'$ between strong endofunctors (F, s) and (G, t) which is compatible with the strengths ('strong natural transformation': see *e.g.* [51]) determines a pseudonatural transformation $\widetilde{\sigma} : \widetilde{F} \Rightarrow \widetilde{G}$ on $\mathbf{Para}(\mathbb{C})$. Each component $(\widetilde{\sigma})_A$ is just $\widetilde{\sigma_A}$, and for a 1-cell $f : P \otimes A \to B$ the 2-cell $\overline{\widetilde{\sigma}}_f$ witnessing naturality is the canonical isomorphism $I \otimes P \xrightarrow{\cong} P \otimes I$.

Because bicategories have a second layer of structure, there is also a notion of map between pseudonatural transformations.

Definition 3.5. A *modification* $\boldsymbol{m} : \eta \to \theta$ between pseudonatural transformations $F \Rightarrow G : \mathscr{B} \to \mathscr{C}$ consists of a 2-cell $\boldsymbol{m}_B : \eta_B \Rightarrow \theta_B$ for every $B \in \mathscr{B}$, subject to an axiom expressing compatibility between $\boldsymbol{m}$ and each $\overline{\eta}_f$ and $\overline{\theta}_f$.

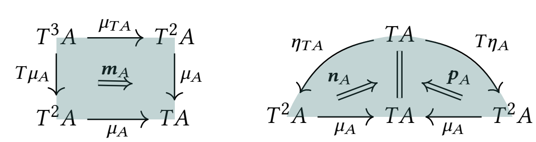
For any bicategories $\mathscr{B}$ and $\mathscr{C}$ there exists a bicategory $\mathrm{Hom}(\mathscr{B}, \mathscr{C})$ with objects pseudofunctors, 1-cells pseudonatural transformations, and 2-cells modifications.

3.2 Pseudomonads and Kleisli bicategories

The bicategorical correlate of a monad is a *pseudomonad*.

Definition 3.6 ([48]). A *pseudomonad* on a bicategory $\mathscr{B}$ consists of a pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ equipped with:

- *Unit* and *multiplication* pseudonatural transformations $\eta : \mathrm{id} \Rightarrow T$ and $\mu : T^2 \Longrightarrow T$, where $T^2 = T \circ T$;
- Invertible modifications $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}$ with components



replacing the usual monad laws, and satisfying two further coherence axioms.

A simple example is given by the Writer pseudomonad on $\mathbf{Cat}$, the bicategory with objects small categories, 1-cells functors, and 2-cells natural transformations. The structural isomorphisms a, l and r are all the identity (giving a *2-category*).

Example 3.7. Let $(\mathbb{C}, \otimes, I)$ be a monoidal category. The pseudofunctor $(-) \times \mathbb{C} : \mathbf{Cat} \to \mathbf{Cat}$ has a pseudomonad structure with 1-cell components

$$\eta_{\mathbb{D}} \;=\; \mathbb{D} \xrightarrow{\cong} \mathbb{D} \times 1 \xrightarrow{\mathbb{D} \times I} \mathbb{D} \times \mathbb{C}$$

$$\mu_{\mathbb{D}} \;=\; (\mathbb{D} \times \mathbb{C}) \times \mathbb{C} \xrightarrow{\cong} \mathbb{D} \times (\mathbb{C} \times \mathbb{C}) \xrightarrow{\mathbb{D} \times \otimes} \mathbb{D} \times \mathbb{C}$$

and 2-cell components $\boldsymbol{m}, \boldsymbol{n}$ and $\boldsymbol{p}$ given by the associator and unitors for the monoidal structure in $\mathbb{C}$.

Example 3.8. Every strong monad (T, μ, η, t) on a monoidal category $(\mathbb{C}, \otimes, I)$ determines a pseudomonad on $\mathbf{Para}(\mathbb{C})$: the underlying pseudofunctor is $\widetilde{T}$ and the pseudonatural transformations are $\widetilde{\mu}$ and $\widetilde{\eta}$ (recall Example 3.4). This remains true if the monoidal structure is replaced by an *action* (as in *e.g.* [61]).

3.3 Monoidal bicategories

A monoidal bicategory is a bicategory equipped with a unit object and a tensor product which is only weakly associative and unital. To motivate the construction, we explain how a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ induces a monoidal structure on $\mathbf{Para}(\mathbb{C})$, with the same action on objects.

The idea is that we can combine the parameters using $\otimes$. For 1-cells $f : P \otimes A \to B$ and $g : P' \otimes A' \to B'$, we set $f \mathbin{\widetilde{\otimes}} g$ to be the object $P \otimes P'$ equipped with

$$(P \otimes P') \otimes (A \otimes A') \xrightarrow{\cong} (P \otimes A) \otimes (P' \otimes A') \xrightarrow{f \otimes g} B \otimes B'$$

where the first map is defined using the symmetry of $\otimes$. On 2-cells, we use the tensor product of maps in $\mathbb{C}$. This construction does not strictly preserve identities and composition, but it does preserve them up to isomorphism. Thus, we get a pseudofunctor $\widetilde{\otimes} : \mathbf{Para}(\mathbb{C}) \times \mathbf{Para}(\mathbb{C}) \longrightarrow \mathbf{Para}(\mathbb{C})$.

We examine the sense in which this tensor is associative and unital, by lifting the structural isomorphisms from $\mathbb{C}$. Every map $f : A \to B$ in $\mathbb{C}$ determines a 1-cell $\widetilde{f}$ in $\mathbf{Para}(\mathbb{C})$ given by the object I and the composite $(I \otimes A \xrightarrow{\cong} A \xrightarrow{f} B)$, where $\cong$ is the unit isomorphism. If f has an inverse f^{-1}, the composite $\widetilde{f} \circ \widetilde{f^{-1}}$ has parameter $I \otimes I$ and thus cannot be the identity. But it is isomorphic to the identity: the pair $(\widetilde{f}, \widetilde{f^{-1}})$ is known as an *equivalence* (an 'isomorphism up to isomorphism'). Thus, although the tensor $\otimes$ on $\mathbb{C}$ is associative and unital up to isomorphism, the tensor $\widetilde{\otimes}$ on $\mathbf{Para}(\mathbb{C})$ is only associative and unital up to equivalence. The structural 1-cells are all pseudonatural in a canonical way (Example 3.4).

Following the general pattern of "bicategorification", the triangle and pentagon axioms of a monoidal category now only hold up to isomorphism: one route round the pentagon has three sides and the other has two, so one composite has parameter $I^{\otimes 3}$ and the other has parameter $I^{\otimes 2}$. These are canonically isomorphic, so we get families of invertible 2-cells witnessing the categorical axioms. All the structure we have defined so far has used the canonical isomorphisms of $\mathbb{C}$, so these families are actually modifications on $\mathbf{Para}(\mathbb{C})$. Moreover, by the axioms of a monoidal category, these structural modifications satisfy axioms of their own.

In summary, a monoidal bicategory is a bicategory equipped with an object I, a pseudofunctor $\widetilde{\otimes}$, pseudonatural families of equivalences witnessing the weak associativity and unitality of $\widetilde{\otimes}$, and invertible modifications witnessing the axioms of a monoidal category. We now make this precise, starting with the definition of equivalences. These generalize equivalences of categories.

Definition 3.9. An *equivalence* between objects A and B in a bicategory $\mathscr{B}$ is a pair of 1-cells $f : A \to B$ and $f^{\bullet} : B \to A$ together with invertible 1-cells $f \circ f^{\bullet} \Rightarrow \mathsf{Id}_B$ and $\mathsf{Id}_A \Rightarrow f^{\bullet} \circ f$.

A *pseudonatural equivalence* is a pseudonatural transformation in which each component has the structure of an equivalence.

The definition is now as advertised. To state it, we introduce some notation for the 2-cell diagrams—known as *pasting diagrams*—that we will use in the rest of the paper.

NOTATION 1. *To save space and improve readability,*

- *We use juxtaposition for the tensor product, e.g. $(AB)C$ means $(A \otimes B) \otimes C$;*
- *We omit the subscripts on the components of pseudonatural transformations and modifications,* e.g. $\boldsymbol{m}$ *instead of* $\boldsymbol{m}_A$*;*
- *We use a subscript notation for the action of a pseudofunctor T,* e.g. T_{AT_B} *means $T(A \otimes T(B))$.*
- *We write $\cong$ for any pseudonaturality 2-cell as in* (5)*, and in equations we omit the arrows showing the directions of 2-cells. These labels can be inferred from the type.*

Definition 3.10 (e.g. *[71]).* A *monoidal bicategory* is a bicategory $\mathscr{B}$ equipped with a pseudofunctor $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ and an object $I \in \mathscr{B}$, together with the following data:

- Pseudonatural equivalences α, λ and ρ with components $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ (the *associator*), $\lambda_A : I \otimes A \to A$, and $\rho_A : A \otimes I \to A$ (the *unitors*);
- Invertible modifications $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{m}$ and $\mathfrak{r}$ with components shown in Figure 2, subject to coherence axioms.

A *symmetric* monoidal bicategory is a monoidal bicategory equipped with a pseudonatural equivalence β with components $\beta_{A,B} : A \otimes B \to B \otimes A$, called the *braiding*, and invertible modifications governing the possible shufflings of three objects and expressing the symmetry of the braiding, subject to coherence axioms.

For example (see *e.g.* [71] for full details), the cartesian product on the category **Set** induces a monoidal structure on the bicategory $\mathbf{Span}(\mathbf{Set})$ introduced in Section 2. The pseudofunctor $\otimes$ is defined on objects as $A \otimes A' = A \times A'$, and for spans $A \leftarrow S \to B$ and $A' \leftarrow S' \to B'$ we take the component-wise product to obtain $A \times A' \leftarrow S \times S' \to B \times B'$.

We also record the outcome of our discussion above; this establishes a conjecture made in [4].

Example 3.11. If $(\mathbb{C}, \otimes, I)$ is a symmetric monoidal category, this lifts to a symmetric monoidal structure on $\mathbf{Para}(\mathbb{C})$.

General point. The coherence axioms of a monoidal bicategory can be difficult to verify directly. However, in many cases of interest the monoidal structure is induced from a more fundamental construction, as in $\mathbf{Span}(\mathbf{Set})$ above. This gives a systematic method for constructing (symmetric) monoidal bicategories: see [80].

3.4 Coherence theorems

As we have seen, bicategorical structures involve considerable data and many equations. Much of the difficulty, however, is tamed by various *coherence theorems.* These generally show that any two parallel 2-cells built out of the structural data are equal. Appropriate coherence theorems apply to bicategories [47] pseudofunctors [26], (symmetric) monoidal bicategories [24, 27] and pseudomonads [41].

We rely heavily on the coherence of bicategories and pseudofunctors when writing pasting diagrams of 2-cells: in particular we omit all compositors and unitors for pseudofunctors, and ignore the weakness of 1-cell composition. Thus, strictly speaking our diagrams do not type-check, but coherence guarantees the resulting

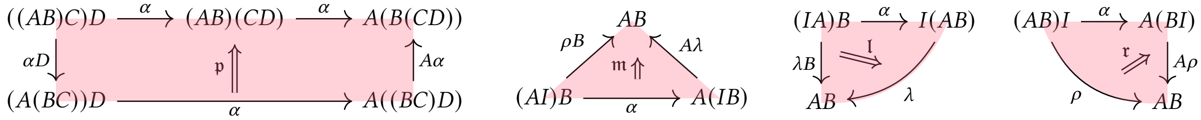


Figure 2: The structural modifications of a monoidal bicategory

2-cell is the same no matter how one fills in the structural details. This is standard practice; for precise justification see *e.g.* [68, §2.2].

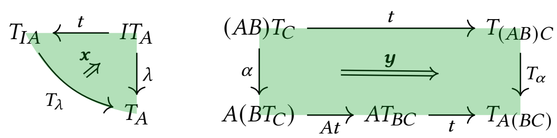
4 STRONG PSEUDOMONADS

We follow the categorical setting by first saying what it means for a pseudo*functor* to be strong, then giving the additional data and axioms to make a pseudo*monad* strong.

4.1 Strong pseudofunctors

For the moment we only consider strengths on the left. In all diagrams below we follow our Notation 1.

Definition 4.1. Let $(\mathscr{B}, \otimes, I)$ be a monoidal bicategory. A *left strength* for a pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ is a pseudonatural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$, equipped with invertible modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ expressing the compatibility of t with the left unitor and the associator:



These modifications must themselves be compatible with the monoidal structure, as per the two axioms of Figure 3.

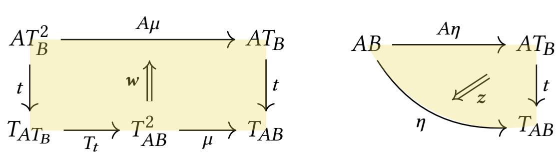
A left strength for a pseudofunctor T can be used to define a parametrised version of the functorial action: for any map $\Gamma \otimes X \to Y$ we can now define a map $\Gamma \otimes TX \to TY$. This suggests the following (recall Example 3.2 and Example 3.4).

Example 4.2. If (F, t) is a strong functor on a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ (see *e.g.* [40, 51]), then the induced pseudofunctor $\widetilde{F}$ on $\mathbf{Para}(\mathbb{C})$ is also strong. The pseudonatural transformation has components $\widetilde{t}_{A,B} := \widetilde{t_{A,B}}$; this has parameter I, so $\boldsymbol{x}$ and $\boldsymbol{y}$ are both of the form $I^{\otimes i} \xrightarrow{\cong} I^{\otimes j}$ for $i, j \in \mathbb{N}$.

4.2 Strong pseudomonads

If a strong pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ is also a pseudomonad, then we must ask for additional data to relate the strength and the monad structure, and this data must be compatible with the modifications $\boldsymbol{x}, \boldsymbol{y}$ we already have.

Definition 4.3. Let $(\mathscr{B}, \otimes, I)$ be a monoidal bicategory. A *left strength* for a pseudomonad (T, η, μ) consists of a left strength $(t, \boldsymbol{x}, \boldsymbol{y})$ for the underlying pseudofunctor, together with invertible modifications



expressing the compatibility of t with the pseudomonad structure. This is subject to two axioms expressing compatibility with the monad structure and two axioms expressing compatibility for $\boldsymbol{x}$ with $\boldsymbol{w}$ and $\boldsymbol{z}$, respectively (Figure 4), and two axioms expressing compatibility for $\boldsymbol{y}$ with $\boldsymbol{w}$ and $\boldsymbol{z}$, respectively (Figure 5).

Extending Example 3.8 and Example 4.2, we obtain the following. The definitions of $\boldsymbol{w}$ and $\boldsymbol{z}$ are similar to those for $\boldsymbol{x}$ and $\boldsymbol{y}$.

Example 4.4. A strong monad on a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ determines a strong pseudomonad on $\mathbf{Para}(\mathbb{C})$.

4.2.1 Note on related work. Strengths for pseudomonads were first defined by Tanaka [75, 76] for applications in categorical universal algebra. We improve on this definition in several ways. We make conceptual progress by cleanly separating strong pseudofunctors from strong pseudomonads. Then we show that only 8 axioms suffice for a coherent definition (Lemma 4.5 below). Finally, in Section 7 we bring a new perspective on pseudostrengths in terms of higher monoidal actions (*c.f.* [22]), obtaining a form of coherence.

In more recent related work, Slattery [70] defines strong (relative) 2-monads on 2-multicategories. An investigation in this direction is important but seems orthogonal to the work presented here.

The details of the two compatibility conditions in the next lemma will not appear in what follows, so we leave them for Appendix A.

Lemma 4.5.
(1) *Given the axioms of Definition 4.3, the modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ are suitably compatible with the monoidal modification* $\mathfrak{l}$.
(2) *Given the axioms of Definition 4.3, the modifications $\boldsymbol{z}$ and $\boldsymbol{w}$ are suitably compatible with the monad modification $\boldsymbol{p}$.*

4.3 Basic examples of strong pseudomonads

In this section we show that several important classes of pseudomonad are strong in the way one would expect from the categorical setting. Many of the proofs essentially come down to the relevant coherence theorem; we give more details in Appendix F.2.

Recall that if (M, m, e) is a monoid in a monoidal category $(\mathbb{C}, \otimes, I)$ then $(-) \otimes M$ becomes a monad with unit and multiplication given via e and m (*c.f.* Example 3.7). This monad is canonically strong, with strength given by the structural isomorphism $A \otimes (B \otimes M) \xrightarrow{\cong} (A \otimes B) \otimes M$. Also note that every monad T is strong with respect to the cocartesian structure $(0, +)$, with strength $[T\mathrm{inl} \circ \eta_A, T\mathrm{inr}] : A + TB \to T(A + B)$. These facts bicategorify. The bicategorical version of a monoid is called a *pseudomonoid* [13, 36], and every pseudomonoid defines a pseudomonad similarly to Example 3.7.

Lemma 4.6.
(1) *For any pseudomonoid (M, m, e, a, l, r) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ the pseudomonad $(-) \otimes M$ has a strength given by the pseudo-inverse $\alpha^\bullet$ of the associator for $\otimes$.*

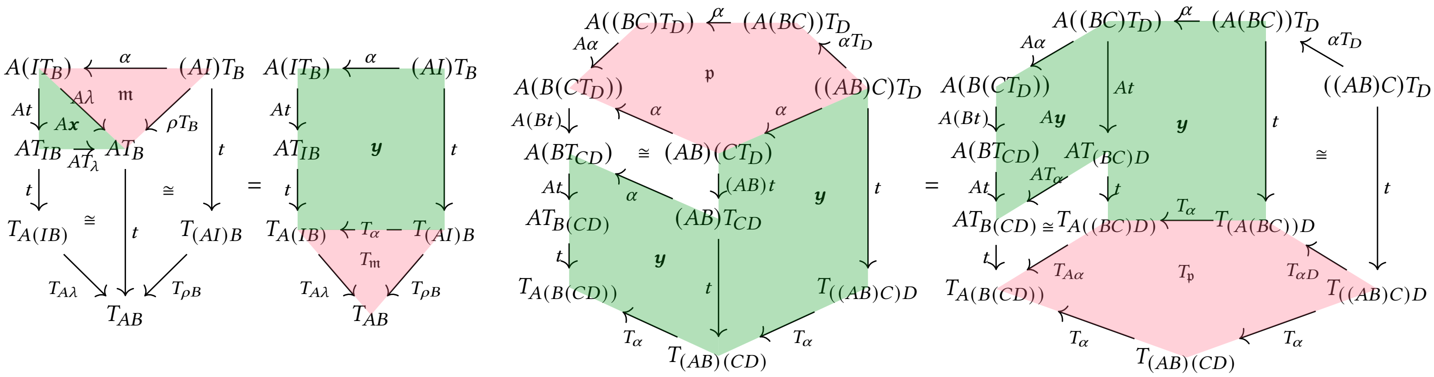


Figure 3: Coherence axioms for a strong pseudofunctor.

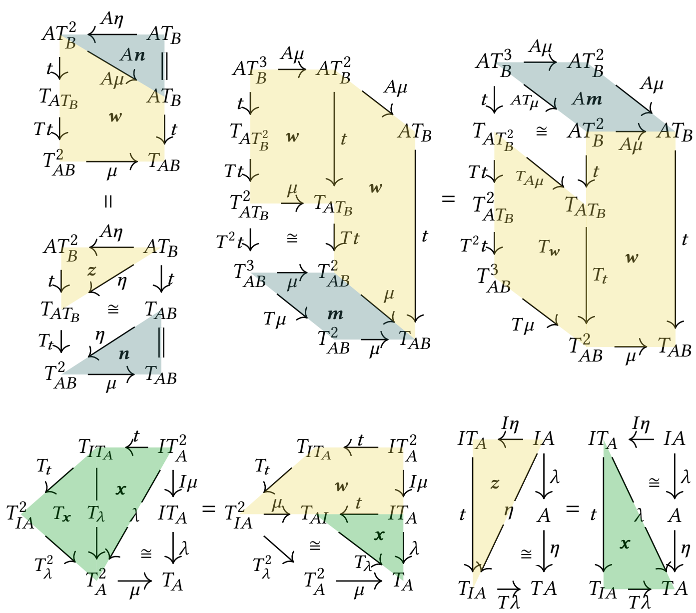


Figure 4: Coherence axioms for a strong pseudomonad: compatibility with the pseudomonad structure, and relating x with z and w.

(2) *Every pseudomonad is canonically strong with respect to the cocartesian monoidal structure* $(+, 0)$.

A pseudomonoid in $(\mathbf{Cat}, \times, 1)$ is exactly a monoidal category, so Lemma 4.6(1) applies in particular to the Writer pseudomonad (Example 3.7). We can also use this lemma to derive a result about pseudomonads on spans. For any category $\mathbb{C}$ with pullbacks there exists a bicategory of spans $\mathbf{Span}(\mathbb{C})$ similar to that defined in Section 2 for $\mathbf{Set}$. For $\mathbb{C} := \mathbf{Set}$, or more generally any *lextensive* category [5], the bicategory $\mathbf{Span}(\mathbb{C})$ has finite biproducts—bicategorical products and coproducts which coincide—by [43, Theorem 6.2]. Moreover, by [28, Corollary A.4], every *cartesian monad* (monad for which the underlying functor preserves pullbacks, and such that every naturality square for μ and η is a pullback square) lifts to a pseudomonad on $\mathbf{Span}(\mathbb{C})$. So we have the following.

Corollary 4.7. *Any cartesian monad on a lextensive category* $\mathbb{C}$ *(such as* $\mathbf{Set}$*) lifts to a strong pseudomonad on* $\mathbf{Span}(\mathbb{C})$

The next example covers two cases of importance in the semantics of programming languages. The proof follows essentially immediately from the corresponding categorical facts and the particularly strong form of coherence enjoyed by cartesian closed bicategories (see [18, Principle 1.3]).

Lemma 4.8. *For any cartesian closed bicategory (see e.g. [17])* $(\mathscr{B}, \times, 1, \Rightarrow)$ *and objects* $S, R \in \mathscr{B}$*, there exist strong pseudomonads* $S \Rightarrow (S \times -)$ *(the* state *pseudomonad) and* $(- \Rightarrow R) \Rightarrow R$ *(the* continuation *pseudomonad).*

For our final class of examples, recall that every functor F on $\mathbf{Set}$ is canonically strong with respect to the cartesian structure, with $t_{A,B} : A \times FB \to F(A \times B)$ defined by $t_{A,B}(a, w) := F(\lambda b\,.\langle a, b\rangle)(w)$, and moreover that the same construction makes every monad on $\mathbf{Set}$ strong [58, Proposition 3.4]. A similar fact holds for bicategories; the statement for pseudomonads was first proved by Tanaka [75].

Proposition 4.9. *Every pseudofunctor (resp. pseudomonad) on* $(\mathbf{Cat}, \times, 1)$ *has a canonical choice of strength.*

5 BISTRONG, COMMUTATIVE, AND MONOIDAL PSEUDOMONADS

Categorically, it is often the case that a monad T supports a strength on both sides, and the two strengths are compatible: T is then called *bistrong* (see *e.g.* [51]). This is the case, for instance, if T has left strength t and the underlying category is symmetric monoidal, because we can construct a right strength using the symmetry β:

$$T(A) \otimes B \xrightarrow{\beta} B \otimes T(A) \xrightarrow{t} T(B \otimes A) \xrightarrow{T\beta} T(A \otimes B). \qquad (6)$$

For a bistrong monad (T, t, s) it makes sense to ask whether the two morphisms below coincide:

$$TA \otimes TB \xrightarrow{t} T(TA \otimes B) \xrightarrow{Ts} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B) \qquad (7)$$

$$TA \otimes TB \xrightarrow{s} T(A \otimes TB) \xrightarrow{Tt} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B) \qquad (8)$$

When they do, T is said to be *commutative* [39, 40]. Kock showed that, in this case, the map $TA \otimes TB \to T(A \otimes B)$ (defined in either way above) gives T the structure of a monoidal monad, and conversely that any monoidal monad is in particular bistrong and commutative.

We now bicategorify these results. We introduce the notion of bistrong pseudomonad in Section 5.1. In Section 5.2 we discuss the

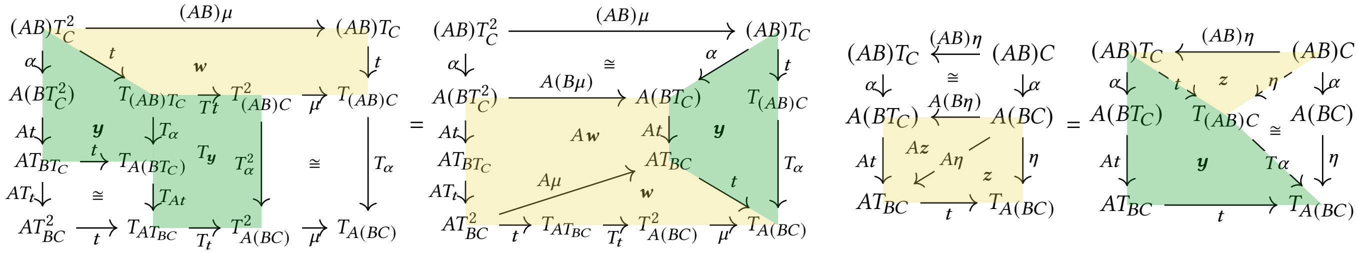


Figure 5: Coherence axioms for a strong pseudomonad: relating y with z and w.

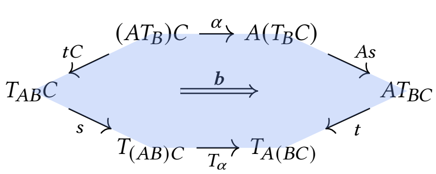
equivalence of commutative and monoidal pseudomonads, which we connect to existing notions due to Hyland & Power [33]. Finally, in Section 5.3 we show the Kleisli bicategory for a bistrong pseudomonad forms a bicategorical version of a well-known model for effectful call-by-value programs.

5.1 Bistrong pseudomonads

A *right strength* for a pseudomonad consists of a pseudonatural transformation $s_{A,B} : T(A) \otimes B \to T(A \otimes B)$ equipped with four invertible modifications analogous to x, y, z, w and satisfying corresponding axioms (we give the data explicitly in Appendix B).

Informally, a left strength $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and a right strength $s_{A,B} : T(A) \otimes B \to T(A \otimes B)$ are *compatible* if parameters on each side can be passed through T in any order. Categorically, one makes this precise by asking that the two obvious maps $(A \otimes TB) \otimes C \to T(A \otimes (B \otimes C))$ are equal. For the bicategorical definition, we replace this equation by a coherent isomorphism.

Definition 5.1. A *bistrong* pseudomonad on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ is a pseudomonad T equipped with a left strength t and a right strength s, and an invertible modification



satisfying the two axioms in Figure 6.

Example 5.2 (Extending Lemma 4.6). If $(\mathscr{B}, \otimes, I)$ is a braided monoidal bicategory and $M \in \mathscr{B}$ has the structure of a *braided pseudomonoid* (see [13]), the pseudomonad $(-) \otimes M$ is canonically bistrong, with s defined using the braiding β and $\boldsymbol{b}$ defined using the pentagonator $\mathfrak{p}$ for $\mathscr{B}$. The axioms follow by coherence [25, 78].

Definition 5.1 is sufficient to recover the categorical situation: if $(\mathscr{B}, \otimes, I)$ is symmetric monoidal and (T, t) is a left-strong pseudomonad, then the composite pseudonatural transformation with components as in (6) can always be given the structure of a right strength for T.

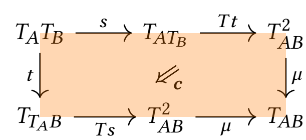
Proposition 5.3. *Every left-strong pseudomonad on a symmetric monoidal bicategory is bistrong in a canonical way.*

Corollary 5.4 (Extending Example 4.4). *If (T, t) is a strong monad on a symmetric monoidal category $(\mathbb{C}, \otimes, I)$, the induced pseudomonad on* $\mathbf{Para}(\mathbb{C})$ *is canonically bistrong.*

5.2 Commutative and monoidal pseudomonads

We now define commutativity for bistrong pseudomonads. Following the usual pattern for bicategorification, the definition is in terms of an invertible 2-cell between the morphisms defined in (7) and (8). Our definition is a straightforward adaptation of Hyland & Power's [33, Definition 5] to the weaker setting of bistrong pseudomonads on a monoidal bicategory.

Definition 5.5. A commutative pseudomonad on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ is a bistrong pseudomonad (T, μ, η, t, s) equipped with an invertible modification



subject to coherence axioms as in [33, Definition 5] (we give these explicitly in Appendix D).

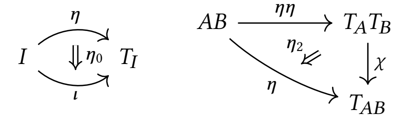
Example 5.6 (Extending Example 5.2). If $(\mathscr{B}, \otimes, I)$ is a symmetric monoidal bicategory and $M \in \mathscr{B}$ has the structure of a *symmetric pseudomonoid* (see [13]), the pseudomonad $(-) \otimes M$ is canonically commutative, with $\boldsymbol{c}$ defined using the braiding on M and symmetric structure on $\mathscr{B}$; the axioms follow by coherence [27, 78].

Example 5.7 (Extending Corollary 5.4). If (T, t) is a commutative monad on a symmetric monoidal category $(\mathbb{C}, \otimes, I)$, the induced pseudomonad on $\mathbf{Para}(\mathbb{C})$ is canonically commutative.

With the axioms of Definition 5.5 we can verify those of a monoidal pseudomonad, and conversely, so Kock's correspondence result ([40, Theorem 2.3]) holds at this level. We begin by defining monoidal pseudomonads. For the definition of monoidal pseudofunctors, transformations, modifications, see [8, 68].

Definition 5.8. A monoidal pseudomonad on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ is a pseudomonad (T, μ, η) with additional structure:

- A 1-cell $\iota : I \to TI$, and pseudonatural transformation $\chi : TA \otimes TB \to T(A \otimes B)$ with three (omitted) invertible modifications making T a monoidal pseudofunctor;
- invertible 2-cells making η a monoidal pseudonatural transformation:



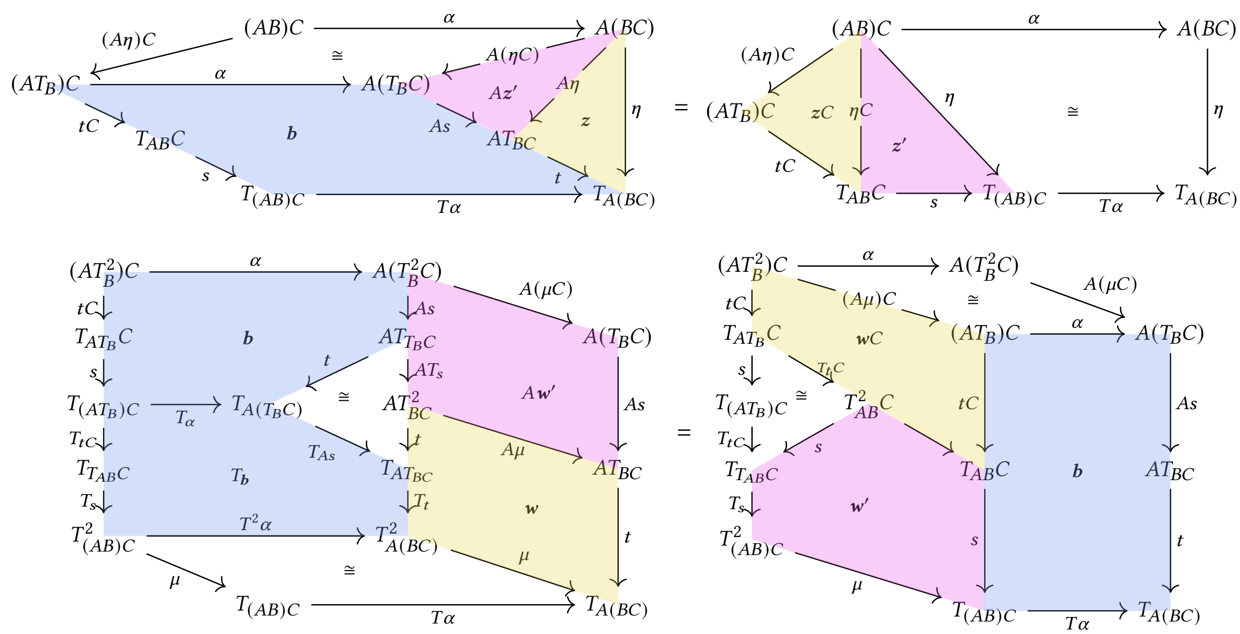
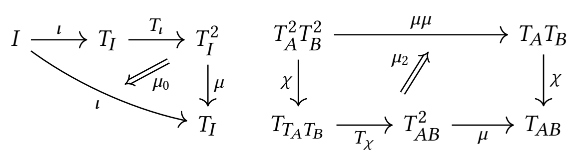


Figure 6: Coherence axioms for a bistrong pseudomonad.

- invertible 2-cells making μ a monoidal pseudonatural transformation:



The pseudomonad modifications $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p})$ must then satisfy the axioms of monoidal modifications and the two pseudomonad laws.

Theorem 5.9. *For any pseudomonad T: every monoidal structure on T canonically induces a commutative structure on T, and every commutative structure on T canonically induces a monoidal structure on T.*

Note that when constructing monoidal structure there is a choice between two isomorphic structures, since our commutativity is only pseudo. The proof is a (long) direct verification; we detail the constructions in Appendix E.

5.3 Premonoidal Kleisli bicategories

A generalisation of Moggi's framework, which does not require a monad explicitly in the syntax, is given by *Freyd categories* [46, 66]. This includes Moggi's approach: the functor $\eta \circ (-) : \mathbb{C} \to \mathbb{C}_T$, which describes the interaction between pure programs (interpreted in $\mathbb{C}$) and effectful ones (interpreted in $\mathbb{C}_T$), forms a Freyd category. In this section we study the Kleisli bicategories associated to the structures discussed above, and show they form *Freyd bicategories* [60]. Thus, the categorical interpretation of call-by-value programs lifts to the bicategorical setting as expected.

Kleisli bicategories. If T is a pseudomonad on a bicategory $\mathscr{B}$, the *Kleisli bicategory* $\mathscr{B}_T$ (*e.g.* [9]) has the same objects as $\mathscr{B}$ and hom-categories $\mathscr{B}_T(A, B) := \mathscr{B}(A, TB)$. The identity on A is the 1-cell $\eta_A \in \mathscr{B}(A, TA)$ and the composition of $f \in \mathscr{B}(A, TB)$ and $g \in \mathscr{B}(B, TC)$ is given by

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu} TC.$$

The structural 2-cells $\mathsf{a}, \mathsf{l}, \mathsf{r}$ in $\mathscr{B}_T$ are constructed using the 2-dimensional structure of the pseudomonad T.

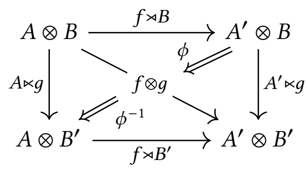
Premonoidal structure. If $\mathscr{B}$ is equipped with a monoidal structure $(\otimes, I)$, then some of this structure is inherited by $\mathscr{B}_T$ when T is strong. More precisely, if T has a left strength t, then for any object $A \in \mathscr{B}$ the mapping

$$B \xrightarrow{f} TB' \quad \longmapsto \quad A \otimes B \xrightarrow{A \otimes f} A \otimes TB' \xrightarrow{t} T(A \otimes B') \qquad (9)$$

can be extended to a pseudofunctor $\mathscr{B}_T \to \mathscr{B}_T$ denoted $A \rtimes -$. Similarly, if T is right-strong, then for every object A we have a pseudofunctor $- \ltimes A : \mathscr{B}_T \to \mathscr{B}_T$.

Proposition 5.10. *For a bistrong pseudomonad (T, s, t) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ the families of pseudofunctors $(-\ltimes A)$ and $(A \rtimes -)$ assemble into a* premonoidal structure *on $\mathscr{B}_T$. Together with the canonical pseudofunctor $\mathscr{B} \to \mathscr{B}_T$, which regards pure morphisms as effectful ones, they determine a* Freyd bicategory.

Monoidal Kleisli bicategories. When the pseudomonad T is commutative, the premonoidal structure on $\mathscr{B}_T$ canonically extends to a (pseudo) monoidal structure. The only missing ingredient is the isomorphism ϕ making $\otimes$ a pseudofunctor of two arguments. One constructs this using c, yielding the interchange law below:



This gives an isomorphism in (3). Next we will consider a generalised setting in which ϕ is not invertible.

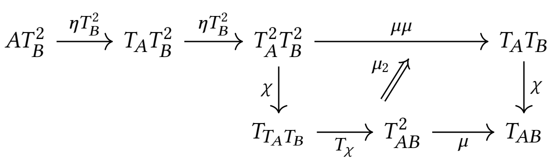
6 CONCURRENT PSEUDOMONADS

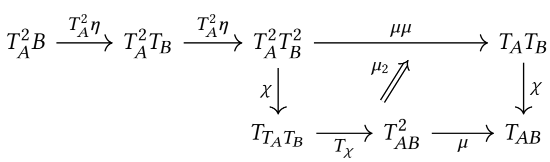
Concurrent pseudomonads illustrate the expressive power of 2-dimensional category theory. Their definition is unequivocally 2-categorical because, for the first time in this paper, we make use of non-invertible 2-cells (and so it would not be sufficient to work with a category 'up to isomorphism', as is commonly done).

6.1 Definition and strength

Definition 6.1. A *concurrent pseudomonad* on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ consists of the same data as a monoidal pseudomonad (Definition 5.8), with axioms modified as follows:

- The modification μ_2 is no longer required to be invertible;
- The composite 2-cells





are now required to be invertible.

The coherence axioms are the same as for a monoidal pseudomonad.

There are likely many examples of this structure.

Example 6.2. For any non-empty set Σ the set of finite strings Σ^* is a monoid in $(\mathbf{Set}, \times, 1)$ and so also in the monoidal category of sets and relations $(\mathbf{Rel}, \times, 1)$. Now, **Rel** is a (degenerate) bicategory with the 2-cells given by the inclusion of relations, and the induced writer pseudomonad $(-) \times \Sigma^*$ is strong but not commutative. It has a concurrent structure with χ defined by

$$(A \times \Sigma^*) \times (B \times \Sigma^*) \to \mathcal{P}((A \times B) \times \Sigma^*)$$
$$(a, u, b, v) \mapsto \{(a, b, w) \mid w \text{ is an interleaving of } u \text{ and } v\}$$

and μ_2 given by the inclusion, which is in general strict.

A careful analysis of the proof of Theorem 5.9 shows that the invertibility of μ_2 is not needed, except where it is precomposed with η as in the definition above. This yields the following:

Proposition 6.3. *Every concurrent pseudomonad has a canonical bistrong structure.*

The next result shows that Definition 6.1 does indeed capture the weak interchange law (3). Lax normal functors are defined like pseudofunctors, without the constraint that the compositor 2-cell ϕ is invertible (*e.g.* [21]).

Proposition 6.4. *For any concurrent pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the families of pseudofunctors $(- \rtimes A)$ and $(A \ltimes -)$ in the premonoidal structure of $\mathscr{B}_T$ assemble into a* lax normal functor $\otimes$ *of two arguments.*

6.2 Illustration in concurrent game semantics

In this section we illustrate concurrent pseudomonads with the continuation pseudomonad from concurrent games [6]. (Game semantics plays no role in this paper outside this section.) Our model is "truly concurrent", in the sense that programs are represented as partially ordered sets of computational events, rather than as sets of possible traces. This makes the concurrent structure of our pseudomonad clear. The model is a simplified, deterministic version of [6].

6.2.1 Event structures. A *(deterministic) event structure* is a partially ordered set of events related by a partial order modelling causal dependency. Formally it is a partial order $(E, \leq_E)$ such that every $e \in E$ depends on finitely many events, *i.e.* the set $\{e' \mid e' \leq_E e\}$ is finite. Thus a finite, down-closed subset of E represents a possible (partial) execution of the concurrent process modelled by E.

A *map of event structures* $(E, \leq_E) \to (D, \leq_D)$ is an injective function $f : E \to D$ such that if $x \subseteq E$ is down-closed, then the image $f\,x$ is also down-closed. The map f can be understood as a simulation of E in D, or in terms of possible execution traces. For example the map in Figure 1 (in which the arrows are a Hasse diagram for $\leq$) is valid because every possible execution of the domain is also an execution in the codomain.

Event structures support a parallel composition operator $E \otimes D$ (sometimes $E \parallel D$), defined as the disjoint union of partial orders.

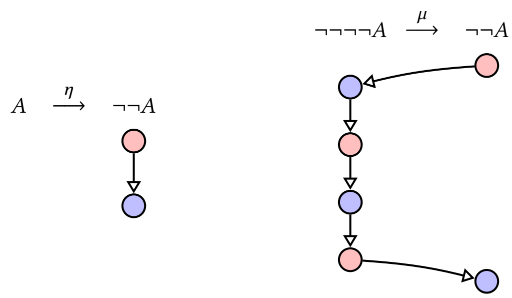
6.2.2 Games and strategies. In what follows we use somewhat informal language to focus on illustrating the concepts. A *game* is an event structure A equipped with a polarity function $A \to \{+, -\}$ assigning "moves" to the program (+) and the environment (−). The game $A^\perp$ swaps the moves of the program and the environment in A: it has the same events, with polarity reversed. A strategy over the game A is an event structure S with a projection map $p : S \to A$ satisfying a lifting condition which plays no role in this section [6].

There is a bicategory of concurrent games $\mathscr{G}$ as follows:

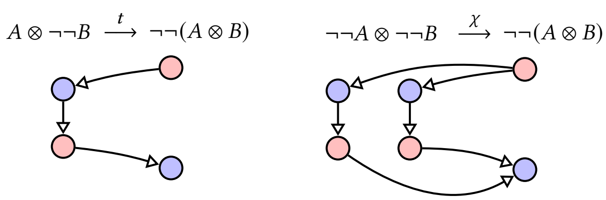
- objects are *negative games*: games whose minimal events are all negative ("the environment always acts first").
- 1-cells from A to B are negative strategies over the game $A^\perp \otimes B$. Intuitively, these encode a program's moves as a function of the environment's behaviour.
- 2-cells from a strategy $p : S \to A^\perp \otimes B$ to a strategy $p' : S' \to A^\perp \otimes B$ are maps of event structures $f : S \to S'$ which commute with the projections.

Strategies are composed using a pullback construction in the category of event structures and maps. (This is only determined up to isomorphism, and therefore is only weakly associative.) The identity on a game A is the *copycat* strategy on $A^\perp \otimes A$, in which every environment move is copied by the program.

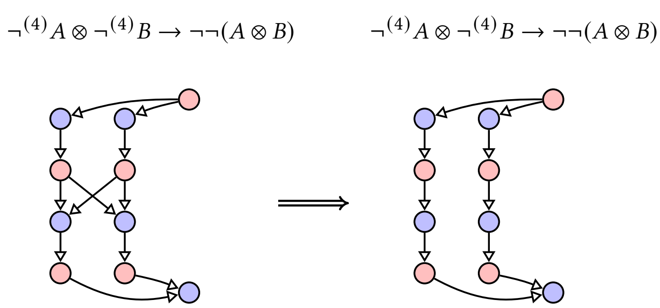
6.2.3 A double-negation concurrent pseudomonad. We can turn a negative game A into a positive game $\neg A$ by appending a single minimal positive move. Similarly we can append a negative move at the beginning of a positive game A' to get a negative game $\neg A'$. The induced operation $\neg\neg$ is a pseudomonad on $\mathscr{G}$, as shown below. (In each diagram, moves of the strategy are positioned underneath the game to which they project. For each strategy we only display the initial portion of appended $\neg$-moves; the rest follows a copycat strategy.)



The effect of the pseudomonad $\neg\neg$ is to track the sequential order of function calls or argument calls. This may be unsurprising: double negation, and game semantics in general, are strongly related to continuation-passing style. This pseudomonad has a strength t. It also has a transformation χ showing that we can represent calls being made in parallel, using the true concurrency of event structures:



However, $\neg\neg$ is not commutative: it is only concurrent. Indeed, one can calculate that the 2-cell $\mu_2 : \mu \circ \neg\neg\chi \circ \chi \Rightarrow \chi \circ (\mu \otimes \mu)$ is the following *non-invertible* map of strategies:



This makes plain the constraints of a midway synchronization point as in the left-hand side of (3), and generalizes the basic example of Figure 1 to a polarized setting.

In summary, game semantics gives a very concrete illustration of a concurrent pseudomonad, in which concurrency is modelled by the true concurrency of event structures. There are more abstract semantics, which we will explore in further work.

7 FORMAL ASPECTS OF STRONG AND MONOIDAL PSEUDOMONADS

A central challenge in developing higher-categorical definitions is choosing the coherence laws on 2-cells. The basic data is usually easy to define: one simply takes the categorical definition and replaces the axioms with invertible 2-cells. But there is no straightforward guide to the axioms on 2-cells (see *e.g.* [68, §2.1]).

In this technical section we justify our definitions in two ways. First, we lift a correspondence between strengths and certain *actions* from the categorical setting (see *e.g.* [51]) to the bicategorical one. This is important from a semantic perspective, but also yields a form of coherence result. Second, we show our definitions arise naturally from higher-categorical considerations. This is a standard approach to verifying the correctness of a definition: *c.f. e.g.* [22, 49]. Together, these provide strong justification for our choice of axioms.

7.1 Strengths as actions

Moggi's *monadic metalanguage* [58] extends the simply-typed λ-calculus with explicit monadic types. It is modelled by a strong monad on a cartesian (more generally, monoidal) category. His *computational λ-calculus*, on the other hand, has the same types as the simply-typed λ-calculus. It is modelled by a Freyd category, which can equivalently be defined as an action extending the monoidal structure (see [45, B.3]). We can see these mathematically capture the same notion of program, because giving a left strength for a monad T on $(\mathbb{C}, \otimes, I)$ is equivalent to giving a left action of $(\mathbb{C}, \otimes, I)$ on the Kleisli category $\mathbb{C}_T$ which extends the monoidal structure (*e.g.* [51, Proposition 4.3]).

We now show this correspondence holds bicategorically. For the definition of bicategorical actions, we use [60, Definition 19]. We first observe that every strong pseudomonad induces an action.

Proposition 7.1. *Every strong pseudomonad (T, t) on $(\mathscr{B}, \otimes, I)$ induces an action of $\mathscr{B}$ on the Kleisli bicategory $\mathscr{B}_T$, where the pseudofunctor $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ is given on objects by $A \triangleright B = A \otimes B$, and on morphisms as*

$$f \triangleright g \;:=\; \left(A \otimes B \xrightarrow{f \otimes g} A' \otimes TB' \xrightarrow{t} T(A' \otimes B')\right)$$

for $f : A \to A'$ and $g : B \to TB'$, with the same action on 2-cells.

The action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ of Proposition 7.1 extends the canonical action $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ given by the monoidal structure. Indeed, we have a pseudonatural transformation

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{B}_T & \xrightarrow{\triangleright} & \mathscr{B}_T \\ {\scriptstyle \mathscr{B}\times K}\uparrow & \overset{\theta}{\nearrow} & \uparrow{\scriptstyle K} \\ \mathscr{B} \times \mathscr{B} & \xrightarrow[\otimes]{} & \mathscr{B} \end{array} \qquad (10)$$

where $K : \mathscr{B} \to \mathscr{B}_T$ is the identity-on-objects pseudofunctor sending $f : A \to A'$ to $\eta_{A'} \circ f : A \to TA'$. Moreover, the two actions $\triangleright$ and $\otimes$ agree on objects, and the 1-cell components $\theta_{A,B}$ of the transformation are all the identity. Such a transformation is known as an *icon* [42]. The 2-cell components of θ are nontrivial: for each $f : A \to A'$ and $g : B \to B'$ we have an isomorphism

$$\theta_{f,g} : f \triangleright K(g) \overset{\cong}{\Longrightarrow} K(f \otimes g)$$

derived from the modification z, satisfying the coherence laws.

We now prove an equivalence between left strengths and left actions. Our correspondence theorem uses the following two categories for a pseudomonad T on $(\mathscr{B}, \otimes, I)$:

- **LeftStr**(T), the category whose objects are left strengths for T, and whose morphisms from t to t' are modifications which commute with all the strength data;
- **LeftExt**(T), the category whose objects are extensions of the canonical action of $\mathscr{B}$ on itself, in the sense they are a *0-strict morphism of actions* as defined in [60], and whose morphisms from $(\triangleright, \theta)$ to $(\triangleright', \theta')$ are icons $\triangleright \Rightarrow \triangleright'$ which commute with θ and θ'.

Theorem 7.2. *For any pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the categories* **LeftStr**(T) *and* **LeftExt**(T) *are equivalent.*

This theorem gives a slick way to prove Proposition 5.10, because constructing an action is easier than constructing the strength. Moreover, Section 5.3 suggests the following extension.

Theorem 7.3. *For any pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, there is an equivalence of categories between monoidal structures on $\mathscr{B}_T$ and commutative structures on T, where in each case morphisms are defined analogously to Theorem 7.2.*

7.1.1 Coherence. Theorems 7.2 and 7.3 entail a form of coherence. We outline the argument for strong pseudomonads, but similar remarks hold for the monoidal case.

First, by Theorem 7.2 every strong pseudomonad is isomorphic to one induced by an action. Because an isomorphism in **LeftStr**(T) commutes with all the data, one can—intuitively speaking—push it to the outside of any pasting diagram. Because it is invertible, this implies that a diagram in a strong pseudomonad commutes if and only if it commutes in the corresponding strong pseudomonad induced by an action. But actions are coherent: every diagram of structural 2-cells commutes (see [60, §4]). It follows that every diagram of structural 2-cells commutes in the induced strong pseudomonad, and hence in the starting strong pseudomonad.

7.2 Strengths as internal pseudomonads

We now place our definitions in a wider mathematical context. We shall show the axioms for strong and monoidal pseudomonads (and hence also for concurrent pseudomonads) arise from standard higher-categorical definitions. It follows that our choice of coherence axioms is canonical.

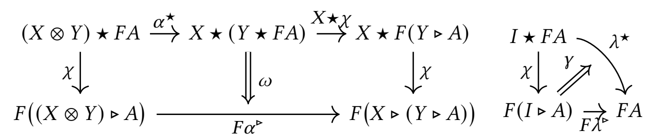
We first recall the 1-dimensional situation. The axioms for strong monads and monoidal monads both arise from the definition of a *monad internal to a 2-category* $\mathscr{C}$. This is defined by taking the categorical definition and replacing the underlying functor T by a 1-cell and the natural transformations μ and η by 2-cells (see *e.g.* [73]). Taking $\mathscr{C} :=$ **Cat** recovers plain monads. Taking the 2-category **MonCat** of monoidal categories, lax monoidal functors, and monoidal natural transformations recovers monoidal monads. For a monoidal category $(\mathbb{V}, \otimes, I)$, taking the 2-category $\mathbb{V}$-**Act** of $\mathbb{V}$-actions, equivariant functors, and equivariant transformations (as defined in *e.g.* [50]) recovers strong monads.

Just as one can define monads in any 2-category, so one can define pseudomonads in any weak 3-category (known as a *tricategory* [24]): see *e.g.* [41]. Our definition of monoidal pseudomonads—and hence concurrent pseudomonads—was carefully chosen to guarantee the following.

Theorem 7.4. *A monoidal pseudomonad such that ι and χ are equipped with the structure of an adjoint equivalence is exactly a pseudomonad internal to the tricategory* **MonBicat** *of monoidal bicategories [8].*

To justify strong pseudomonads we need to work a little harder, because we cannot rely on a pre-existing tricategory of actions. However, for any monoidal bicategory $(\mathscr{V}, \otimes, I)$ we can define a tricategory $\mathscr{V}$-**Act** by small adjustments to the definition of **MonBicat**. We sketch the definitions, reserving more details for Appendix C.

The objects of $\mathscr{V}$-**Act** are left $\mathscr{V}$-actions. The 1-cells $(\triangleright, \alpha^{\triangleright}, \lambda^{\triangleright}) \to (\star, \alpha^{\star}, \lambda^{\star})$ are *equivariant morphisms*, which consist of a pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ between the bicategories acted on, a pseudonatural transformation χ with components $\chi_{X,B} : X \star FB \to F(X \triangleright B)$, and invertible modifications ω and γ with components as shown below, subject to an associativity law and two unit laws:



The 2-cells $(F, \chi, \omega, \gamma) \to (F', \chi', \omega', \gamma')$ are *equivariant transformations*, consisting of a pseudonatural transformation $\sigma : F \Rightarrow F'$ and an invertible modification Γ with components $\Gamma : \chi' \circ (X \star \sigma) \Rightarrow \sigma \circ \chi$ subject to an associativity law and unit law. The 3-cells $(\sigma, \Gamma) \to (\sigma', \Gamma')$ are *equivariant modifications*, which are modifications $q : \sigma \to \sigma'$ subject to a law relating Γ and Γ'.

Because the data and axioms is similar to that for **MonBicat**, it is relatively easy to show $\mathscr{V}$-**Act** forms a tricategory (*c.f.* [8]).

In the 1-dimensional setting a $\mathbb{V}$-action on a $\mathbb{C}$ is equivalently a strong monoidal functor $\mathbb{V} \to [\mathbb{C}, \mathbb{C}]$ into the strict monoidal category of endofunctors on $\mathbb{C}$. We verify our definition of $\mathscr{V}$-**Act** with a bicategorical version of this result. To state the proposition, we restrict to equivariant data that strictly preserves the base bicategories: write **LAct**$(\mathscr{B})$ for the bicategory with objects $\mathscr{V}$-actions on $\mathscr{B}$, 1-cells equivariant morphisms with underlying pseudofunctor $\mathrm{id}_{\mathscr{B}}$, and 2-cells equivariant transformations of the form (id, Γ).

Proposition 7.5. *For any monoidal bicategory $(\mathscr{V}, \otimes, I)$ and bicategory $\mathscr{B}$, the currying biequivalence of [74, §1.34] lifts to a biequivalence between* **LAct**$(\mathscr{B}) \simeq$ **MonBicat**$(\mathscr{V}, \mathrm{Hom}(\mathscr{B}, \mathscr{B}))$.

We can now see that strong pseudomonads have a canonical status:

Theorem 7.6. *A strong pseudomonad on $\mathscr{V}$ is equivalently a pseudomonad on the canonical action of $\mathscr{V}$ on itself in* $\mathscr{V}$-**Act**.

8 CONCLUSION

In this paper we have laid the foundations for modelling effectful programs in 2-dimensional categories. We have introduced a bicategorical version of Moggi's unifying framework (Sections 4 and 5) and seen how the extra structure available in this setting captures phenomena that are otherwise invisible (Section 6). In doing so, we have brought together observations in concurrency theory (*c.f.* [31, 55]) with new kinds of models motivated by entirely different concerns (*e.g.* [6, 11, 16]). Finally, we have laid the basis for further mathematical investigation by showing our definitions arise as expected from purely category-theoretic concerns (Section 7).

Moggi's framework paved the way for understanding effectful programming in many new directions (*e.g.* [34, 37, 53, 62]). We see this paper as the starting point for a fruitful line of future work, mirroring the development that followed Moggi's work. Syntactically, it would be natural to develop the internal languages of the various pseudomonad structures presented here (*c.f.* [17]). Semantically, the development of Section 6 suggests making explicit the 2-dimensional structure implicit in long-standing models such as those detailed in Section 1.1. This includes studying simple cases such as Example 6.2; similar examples arise naturally in areas such as probabilistic programming. It is likely that seeing the full 2-dimensional picture will highlight new connections and theoretical insights. Other forms of model, such as *dialogue categories* [52, 56], are also likely to have natural bicategorical counterparts. More abstractly, the exact structure of the Kleisli bicategory of a concurrent pseudomonad remains to be understood (*c.f.* [21]), as are the connections between "graded pseudomonads", strong pseudomonads, and pseudo-distributive laws (*e.g.* [79]).

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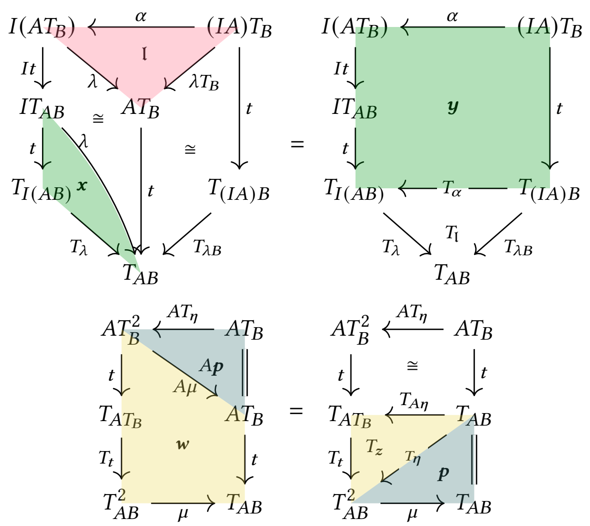
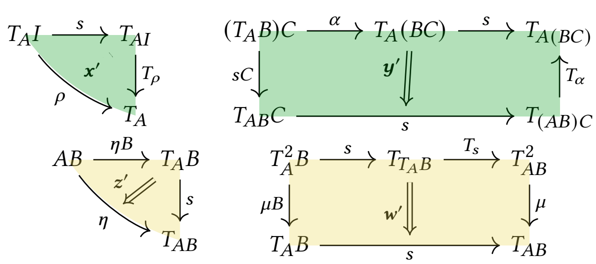
A TWO REDUNDANT AXIOMS FOR LEFT-STRONG PSEUDOMONADS

The two equations below are derivable from the 8 presented in the main body.



B THE DATA FOR RIGHT-STRONG PSEUDOMONADS

The data of a right-strong pseudomonad is shown below. The 8 axioms are essentially those of a left-strong pseudomonad, with the action of parameters on the left replaced by the the corresponding action on the right so that, for example, IT_A is replaced by $T_A I$ and λ is replaced by ρ.

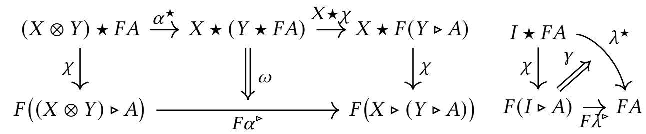


C THE TRICATEGORY OF $\mathscr{V}$-ACTIONS

We give the coherence axioms for the tricategory $\mathscr{V}$-**Act**. Throughout this section we fix a monoidal bicategory $(\mathscr{V}, \otimes, I)$ with left $\mathscr{V}$-actions $(\triangleright, \alpha^{\triangleright}, \lambda^{\triangleright}) : \mathscr{V} \times \mathscr{B} \to \mathscr{B}$ and $(\star, \alpha^{\star}, \lambda^{\star}) : \mathscr{V} \times \mathscr{C} \to \mathscr{C}$ defined as in [60]. Composition and identities are defined using the definition of composition of pseudonatural transformations and modifications. Similarly, the structural transformations and structural modifications are all defined by endowing the corresponding structure in **Bicat** with equivariant structure in the obvious way. In each case, most of the work lies in showing the coherence conditions still hold; the various axioms hold because they hold in **Bicat**.

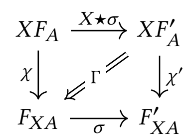
Definition C.1. An *equivariant morphism* $(\triangleright, \alpha^{\triangleright}, \lambda^{\triangleright}) \to (\star, \alpha^{\star}, \lambda^{\star})$ consists of:

(1) A pseudofunctor $F : \mathscr{B} \to \mathscr{C}$;
(2) A pseudonatural transformation $\chi_{X,B} : X \star FB \to F(X \triangleright B)$;
(3) Invertible modifications ω and γ as shown below, subject to the associativity law and two unit laws in Figure 7:



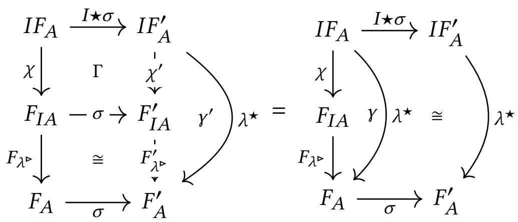
Definition C.2. An *equivariant 2-cell* $(F, \chi, \omega, \gamma) \to (F', \chi', \omega', \gamma')$ between action morphisms of type $(\triangleright, \alpha^{\triangleright}, \lambda^{\triangleright}) \to (\star, \alpha^{\star}, \lambda^{\star})$ consists of:

(1) A pseudonatural transformation $\sigma : F \Rightarrow F'$;
(2) An invertible modification Γ with components as shown:

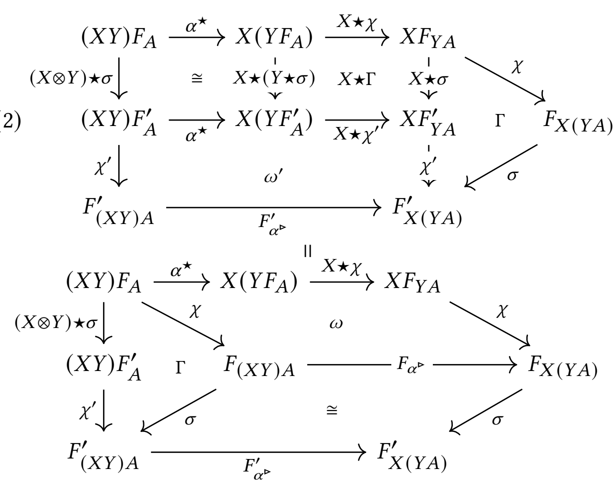


subject to the following unit and associativity laws:

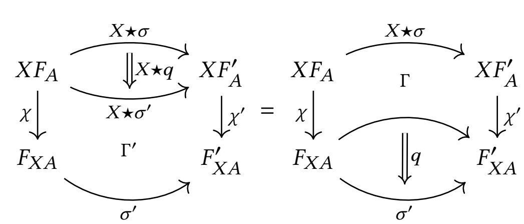
(1)



(2)



Definition C.3. Let $(F, \chi, \omega, \gamma) \to (F', \chi', \omega', \gamma') : (\triangleright, \alpha^{\triangleright}, \lambda^{\triangleright}) \to (\star, \alpha^{\star}, \lambda^{\star})$ be equivariant 1-cells related by equivariant 2-cells $(\sigma, \Gamma), (\sigma', \Gamma') : (F, \chi, \omega, \gamma) \to (F', \chi', \omega', \gamma')$. An *equivariant 3-cell* $(\sigma, \Gamma) \to (\sigma', \Gamma')$ consists of a modification $q : \sigma \to \sigma'$ such that



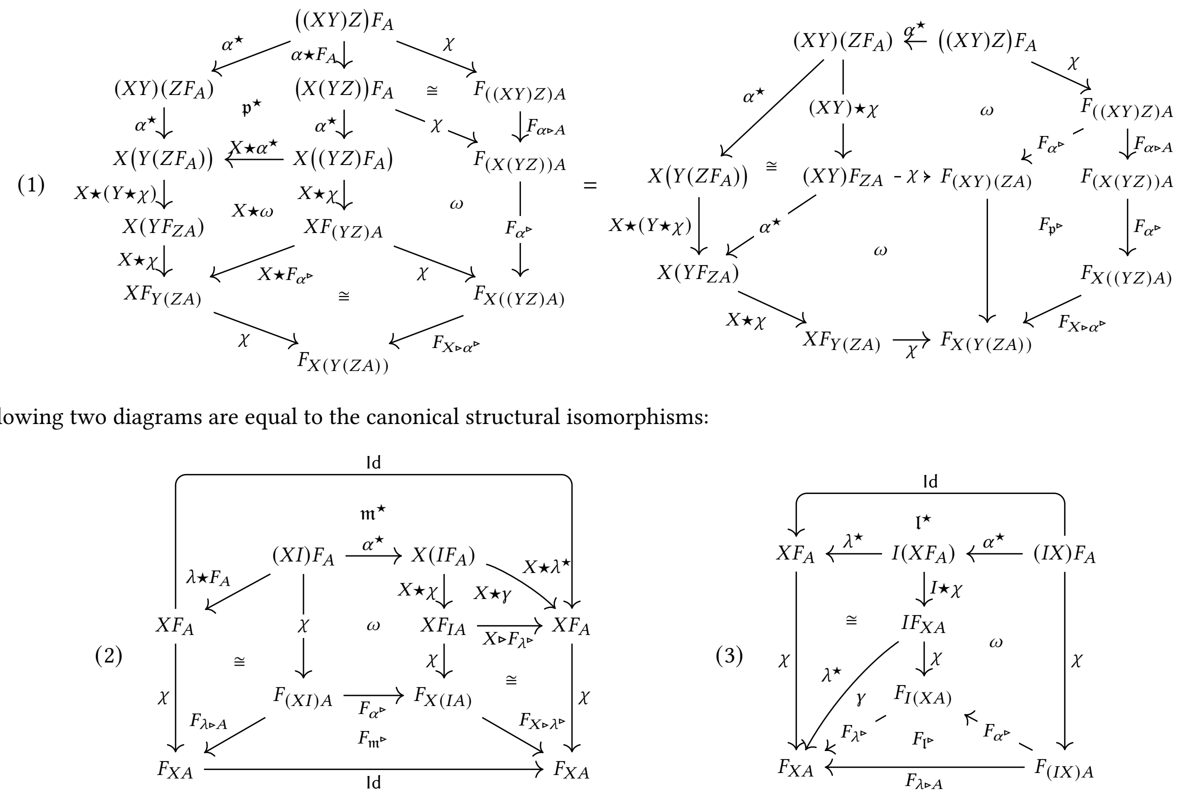


Figure 7: Axioms for equivariant morphisms.

D COHERENCE AXIOMS FOR COMMUTATIVE PSEUDOMONADS

We collect the axioms for the commutativity modification $\boldsymbol{c}$ of a commutative pseudomonad. These are obtained from Hyland & Power's axioms [33] by explicitly adding in the structural isomorphisms of the monoidal bicategory and the bistrong pseudomonad. The axioms are shown in Figures 8 and 9. We have numbered them so they match [33, Definition 5].

E RELATING COMMUTATIVE, CONCURRENT AND MONOIDAL PSEUDOMONADS

We outline the construction of a commutative pseudomonad from a monoidal pseudomonad (Appendix E.1), and the construction of a monoidal pseudomonad from a commutative pseudomonad (Appendix E.2). In doing so, we see how to construct a bistrong pseudomonad from a concurrent pseudomonad (Appendix E.1). Here we just show how to construct the data. The axioms are all checked directly: this is long-winded, but relatively straightforward.

E.1 From monoidal to commutative

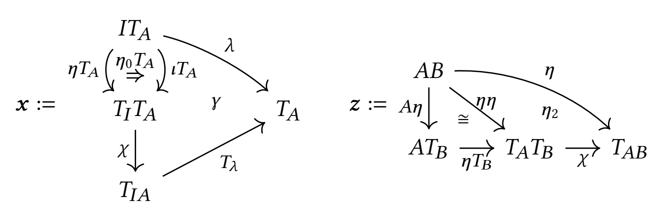
Fix a monoidal pseudomonad as in Definition 5.8, with the three modifications of the underlying monoidal pseudofunctor denoted by γ, δ and ϖ as in [8].

We give the data for a commutative pseudomonad. In doing so we need to construct bistrong structure; because we only use the invertibility of μ_2 in the definition of $\boldsymbol{c}$, this also shows how to construct a *lax* bistrong pseudomonad from a *lax* monoidal pseudomonad. A short check then shows that the invertibility conditions of a concurrent pseudomonad (Definition 6.1) suffice to make all the modifications for the induced bistrong pseudomonad invertible.

For the two strengths, we take:

$$t_{A,B} := AT_B \xrightarrow{\eta T_B} T_A T_B \xrightarrow{\chi} T_{AB}$$
$$s_{A,B} := T_A B \xrightarrow{T_A \eta} T_A T_B \xrightarrow{\chi} T_{AB}$$

We give the structural modifications making the pseudomonad (T, μ, η) left strong; the ones for the right strength are very similar. First, the unit laws:



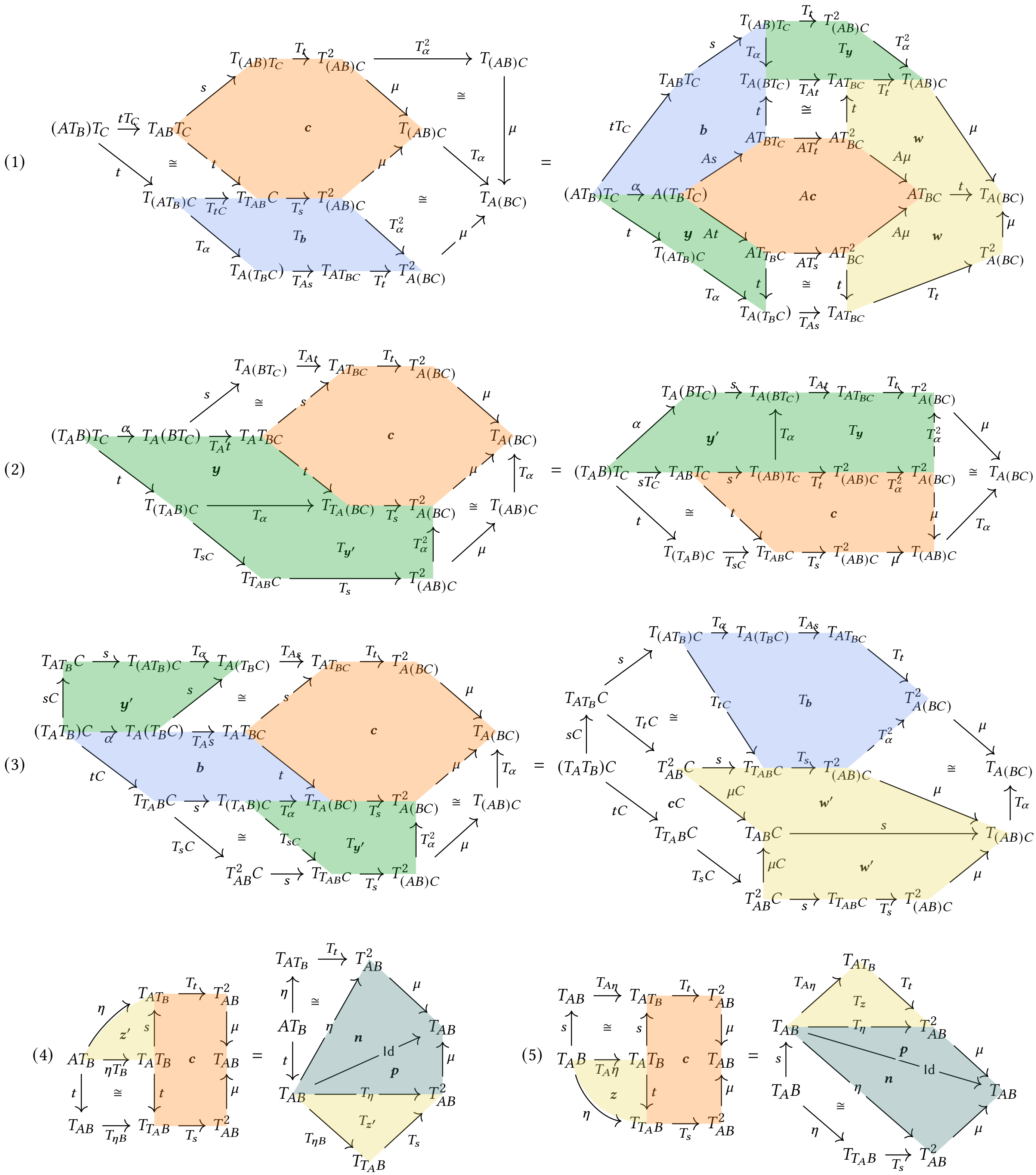


Figure 8: Axioms (1)–(5) for a commutative pseudomonad.

(6)

(7)

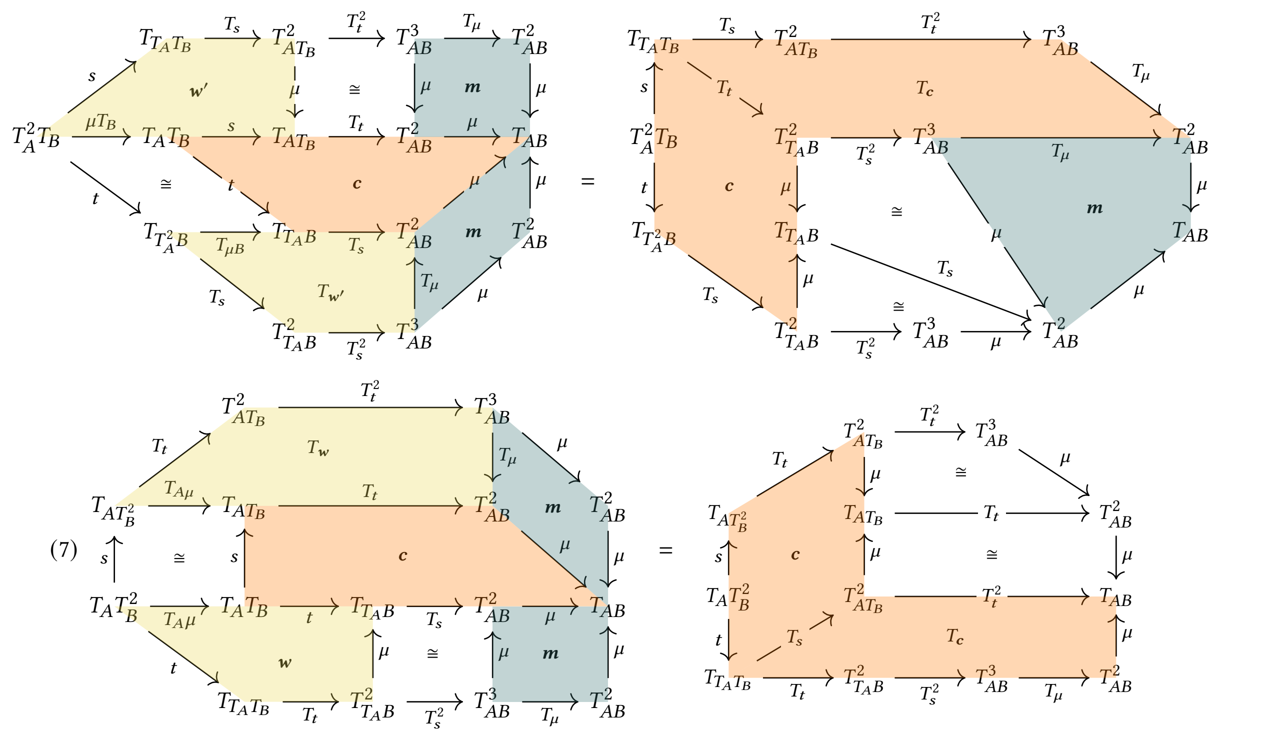
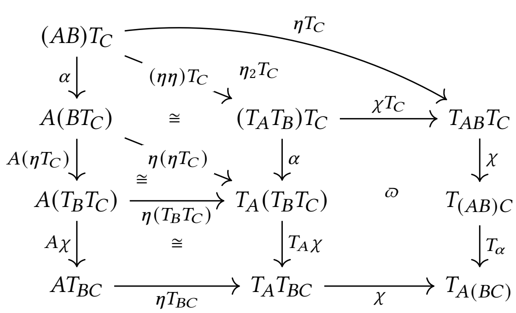


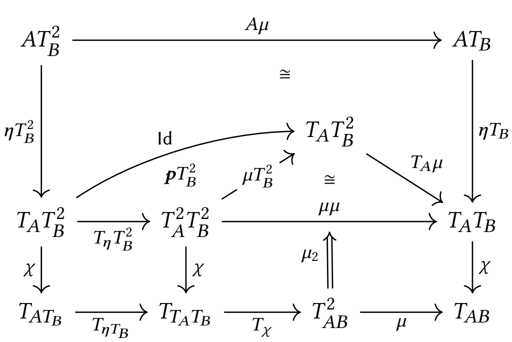
Figure 9: Axioms (6) & (7) for a commutative pseudomonad.

Next, we give the two associativity laws. Note that if T is concurrent, $\boldsymbol{w}$ is invertible even though μ_2 is not.

$\boldsymbol{y} :=$

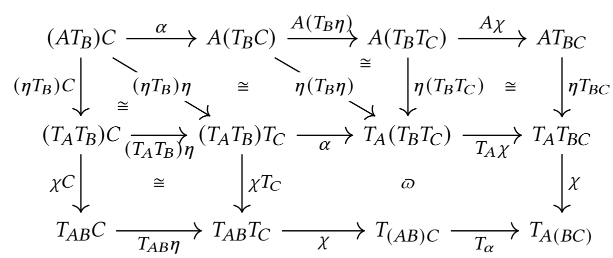


$\boldsymbol{w} :=$



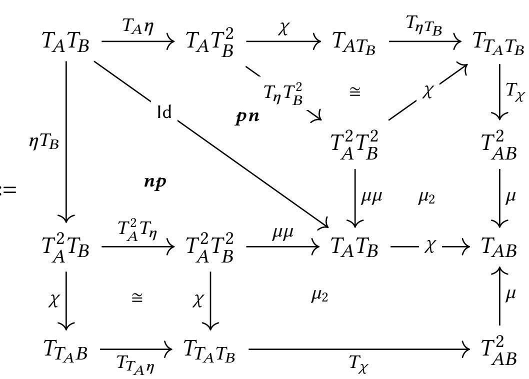
For the bistrong structure:

$\boldsymbol{b} :=$



Finally, for the commutative structure we take the following. Note that this is the only pasting diagram that uses both μ_2 and its inverse.

$\boldsymbol{c} :=$



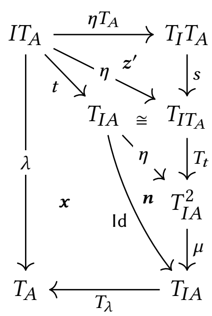
E.2 From commutative to monoidal

Fix a commutative pseudomonad as in Definition 5.5, with right-strong data denoted as in Appendix B. We define a monoidal pseudomonad structure by taking ι to be the identity and

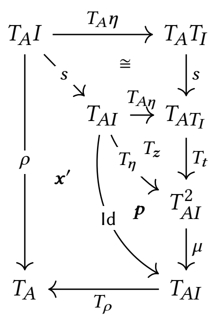
$$\chi := (T_A T_B \xrightarrow{s} T_{AT_B} \xrightarrow{T_t} T^2_{AB} \xrightarrow{\mu} T_{AB})$$

We may therefore take η_0 to be the identity. The modifications ϖ, γ and δ making T a monoidal pseudofunctor as in [68, Definition 2.5] are then defined as follows.

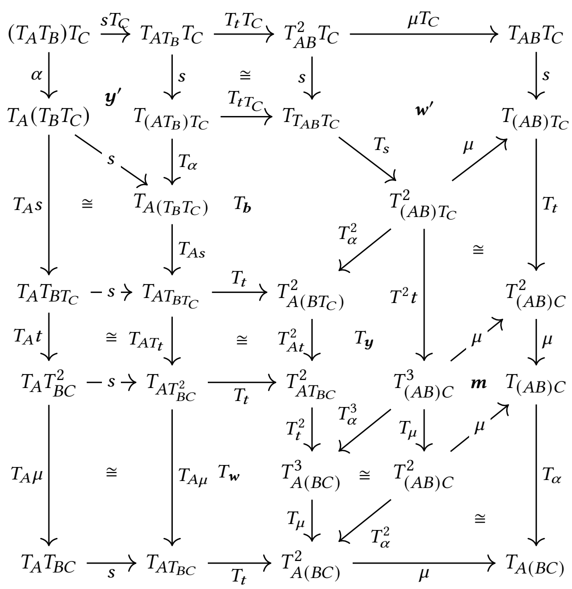
$\gamma :=$



$\delta :=$

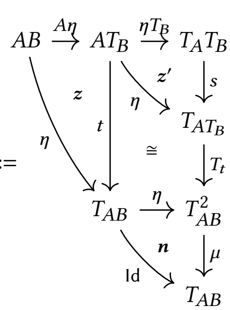


$\varpi :=$

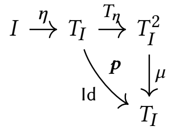


Finally, for the modifications η_2, μ_0 and μ_2 we take:

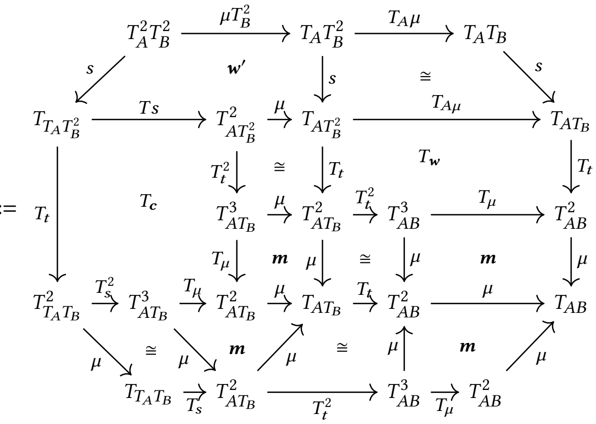
$\eta_2 :=$



$\mu_0 :=$



$\mu_2 :=$



F SKETCHES OF PROOFS OMITTED FROM THE MAIN BODY

F.1 Proofs for Section 4.1

Lemma 4.5. *(1) Given the axioms of Definition 4.3, the modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ are suitably compatible with the monoidal modification $\mathfrak{l}$.*

(2) Given the axioms of Definition 4.3, the modifications $\boldsymbol{z}$ and $\boldsymbol{w}$ are suitably compatible with the monad modification $\boldsymbol{p}$.

Proof. The first of these axioms is proved using that $\mathfrak{l}$ is completely determined by the other structural data of a monoidal bicategory (see [26, p. 64]). For the second axiom, while $\boldsymbol{p}$ is not determined by the rest of the data, the composite $\mu \circ \boldsymbol{p}_T$ is uniquely expressible using $\boldsymbol{n}$ and $\boldsymbol{m}$. This suffices for the proof. □

F.2 Proofs for Section 4.3

Lemma 4.6.

1. *For any pseudomonoid (M, m, e, a, l, r) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ the pseudomonad $(-) \otimes M$ has a strength given by the pseudo-inverse $\alpha^\bullet$ of the associator for $\otimes$.*
2. *Every pseudomonad is canonically strong with respect to the cocartesian monoidal structure $(+, 0)$.*

Proof. For both claims, one constructs the data by following the corresponding 1-categorical argument and filling the commuting diagrams with the appropriate 2-cells; the equations hold by coherence. For (1), for instance, the structural modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ are given using $\mathfrak{l}$ and $\mathfrak{p}$, respectively, while $\boldsymbol{w}$ and $\boldsymbol{z}$ are given using $\mathfrak{r}$ and $\mathfrak{p}$, respectively, together with the pseudonaturality of $\alpha^\bullet$. The axioms hold by the coherence of pseudomonoids [41]. Similarly for (2): the strength has components $[T\mathrm{inl} \circ \eta_A, T\mathrm{inr}] : A + TB \to T(A+B)$ and the structural modifications are given by taking the categorical proof and filling in the commuting diagrams with the appropriate 2-cells. The equations hold by coherence for bicategories with finite products [63] and the fact all the structural 2-cells are invertible. □

Proposition 4.9. *Every pseudofunctor (resp. pseudomonad) on $(\mathbf{Cat}, \times, 1)$ has a canonical choice of strength.*

Proof. Similarly to the categorical proof, for any pseudofunctor $F : \mathbf{Cat} \to \mathbf{Cat}$, and $a \in \mathbb{A}$ one has $F(\lambda b\,.\langle a, b\rangle) : F(\mathbb{B}) \to F(\mathbb{A}\times\mathbb{B})$.

However, since F is now a pseudofunctor we also have a natural transformation $F(\lambda b\,.\langle f, b\rangle)$ for each $f : a \to a'$ in $\mathbb{A}$, with components $F(\lambda b\,.\langle f, b\rangle)_w : F(\lambda b\,.\langle a, b\rangle) \to F(\lambda b\,.\langle a', b\rangle)$ in $F(\mathbb{A}\times\mathbb{B})$. We may therefore define a functor $t_{\mathbb{A},\mathbb{B}} : \mathbb{A} \times F\mathbb{B} \to F(\mathbb{A}\times\mathbb{B})$ sending a pair of objects (a, w) to $F(\lambda b\,.\langle a, b\rangle)(w)$ and a pair of morphisms $(a \xrightarrow{f} a', w \xrightarrow{g} w')$ to the composite $F(\lambda b\,.\langle a', b\rangle)(g) \circ F(\lambda b\,.\langle f, b\rangle)_w$. This is functorial because F is functorial on natural transformations and $F(\lambda b\,.\langle a, b\rangle)$ is a functor, and pseudonatural via the compositor for F.

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined using the compositor and unitor for F, and the coherence of pseudomonads ensures the axioms hold. Finally, if T is a pseudomonad then one defines $\boldsymbol{w}$ and $\boldsymbol{z}$ using the pseudonaturality of η and μ: this is similar to the proof in the categorical setting, where one uses the naturality of the unit and multiplication to show the two compatibility laws hold. Again, the axioms follow from coherence. □

F.3 Proofs for Section 5.3

Proposition 5.10. *For a bistrong pseudomonad (T, s, t) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ the families of pseudofunctors $(-\ltimes A)$ and $(A\rtimes-)$ assemble into a* premonoidal structure *on $\mathscr{B}_T$. Together with the canonical pseudofunctor $\mathscr{B} \to \mathscr{B}_T$, which regards pure morphisms as effectful ones, they determine a* Freyd bicategory.

Proof. We use Theorem 7.2. The binoidal structure is as in (9). Then, the proof consists in constructing a compatible pair of a left action and a right action, where "compatible" means that the two associators coincide on 1-cells, and the structural 2-cells $\widetilde{\mathfrak{p}}$ and $\widetilde{\mathfrak{m}}$ coincide. All of this is verified directly, based on the construction of the actions in Theorem 7.2.

The only remaining difficulty is the pseudonaturality of α in its middle argument, since this is not required for either of the actions. This is where the 2-cell given by the bistrong structure is used. □

F.4 Proofs for Section 7.1

Proposition 7.1. *Every strong pseudomonad (T, t) on $(\mathscr{B}, \otimes, I)$ induces an action of $\mathscr{B}$ on the Kleisli bicategory $\mathscr{B}_T$, where the pseudofunctor $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ is given on objects by $A \triangleright B = A \otimes B$, and on morphisms as*

$$f \triangleright g \;:=\; \left(A \otimes B \xrightarrow{f\otimes g} A' \otimes TB' \xrightarrow{t} T(A' \otimes B')\right)$$

for $f : A \to A'$ and $g : B \to TB'$, with the same action on 2-cells.

Proof. The action on 2-cells is the same as that on morphisms. The compositor and unitor for $\triangleright$ are given by the modifications $\boldsymbol{w}$ and $\boldsymbol{z}$ that come with the strength. We use $\boldsymbol{x}$ and $\boldsymbol{y}$ to construct the strength data $\widetilde{\lambda}$ and $\widetilde{\alpha}$ from the monoidal data λ and α, and finally we use $\boldsymbol{z}$ again to lift $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ to $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{m}}, \widetilde{\mathfrak{l}}$. The strength axioms ensure that this forms an action. □

Theorem 7.2. *For any pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the categories* **LeftStr**(T) *and* **LeftExt**(T) *are equivalent.*

Proof. The proof follows the categorical construction (see [51, Prop. 4.3]). For every left strength t for T, the induced action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ extends the canonical action of $\mathscr{B}$ on itself, by construction. Conversely, any extension $(\triangleright, \theta)$ induces a strength $t_{A,B} = \mathrm{id}_A \triangleright \mathrm{id}_{TB}$, where id_{TB} is regarded as an element of $\mathscr{B}_T(TB, B)$. These constructions are inverses, up to isomorphism, as we verify directly. We then verify the axioms. In each direction, there is a tight match-up between the equations of the given structure and the equations for the required structure; for an outline, see Table 1. □

Strength	Action
Axioms for a strong pseudofunctor (Fig. 3)	Modification axioms for the 2-cells $\widetilde{\mathfrak{m}}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{p}}$ determined by the action extension
Compatibility between **m**, **n**, **p** and **z**, **w**	Pseudofunctor axioms for $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$
Compatibility between **x** and **z**, **w**	Pseudonaturality of the transformation $\widetilde{\lambda}$ determined by the action extension
Compatibility between **y** and **z**, **w**	Pseudonaturality of the transformation $\widetilde{\alpha}$ determined by the action extension

Table 1: Relating the data and equations on each side of the correspondence in Theorem 7.2.