PREMONOIDAL BICATEGORIES

(joint work with Hugo Paquet)

WHY PREMONOIDAL BICATEGORIES

premonoidal categories + bicategorical models

premonoidal categories → axiomatises many semantic models

bicategorical models → more refined / intensional information

eg// spans, games, Prof-based models, graded monads

WHY PREMONOIDAL BICATEGORIES

premonoidal categories

↓

axiomatises many semantic models

+

↓

framework for 2-dimensional semantics

bicategorical models

↓

more refined / intensional information

eg// spans, games, Prof-based models, graded monads

PREMONOIDAL CATEGORIES

└ semantic models of effectful programs

HOW SHOULD WE MODEL EFFECTFUL COMPUTATIONS?

value

Moggi: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to T[\![A]\!]$

for T a strong monad

$st : A \times TB \to T(A \times B)$

effect

eg// WRITER monad $(-) \times \mathcal{L}$

value

$[\![\Gamma \vdash M : A]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!] \times \mathcal{L}$

string to print

HOW SHOULD WE MODEL EFFECTFUL COMPUTATIONS?

<u>Moggi</u>: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to T[\![A]\!]$
for T a strong monad

value

effect

$st : A \times TB \to T(A \times B)$

$$\left[\!\!\left[\frac{x : A \vdash M : B \qquad \vdash N : A}{\vdash \text{let } x = N \text{ in } M : B} \right]\!\!\right] = 1 \xrightarrow{[\![N]\!]} T[\![A]\!] \xrightarrow{[\![M]\!]^\#} T[\![B]\!]$$

Kleisli extension

Moggi: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to T[\![A]\!]$ for T a strong monad

Moggi: $[\![\Gamma \vdash M : A]\!]$ is $\underbrace{\text{an arrow } [\![\Gamma]\!] \to T[\![A]\!] \text{ for } T \text{ a strong monad}}$

||

an arrow in the Kleisli category $\mathbb{C}_T$

<u>Moggi</u>: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to [\![A]\!]$ in the Kleisli category $\mathbb{C}_T$ for T strong

Moggi: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to [\![A]\!]$ in the Kleisli category $\mathcal{C}_T$ for T strong

WORKS WELL FOR let:

$$\left[\!\!\left[\frac{x : A \vdash M : B \qquad \vdash N : A}{\vdash \mathsf{let}\ x = N\ \mathsf{in}\ M : B} \right]\!\!\right] = \underbrace{1 \xrightarrow{[\![N]\!]} [\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!]}_{\text{let-binding directly modelled as composition}}$$

Moggi: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to [\![A]\!]$ in the Kleisli category $\mathcal{C}_T$ for T strong

WHAT ABOUT PRODUCTS?

idea: run $[\![M_1]\!]$ to a value, run $[\![M_2]\!]$ to a value, return the pair

$$\left[\!\!\left[\frac{x : A \vdash M_1 : B_1 \qquad x : A \vdash M_2 : B_2}{x : A \vdash \langle M_1, M_2 \rangle : B_1 \times B_2} \right]\!\!\right]$$

$$= [\![A]\!] \xrightarrow{\langle [\![M_1]\!], [\![M_2]\!] \rangle} T[\![B_1]\!] \times T[\![B_2]\!] \underbrace{\longrightarrow}_{\text{built from strength}} T([\![B_1]\!] \times [\![B_2]\!])$$

<u>Moggi</u>: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to [\![A]\!]$ in the Kleisli category $\mathcal{C}_T$ for T strong

syntax should be free model!

WHAT ABOUT THE MONAD?

if we have function types + unit, $(1 \to -)$ is a strong monad

} what about if we don't?

<u>Moggi</u>: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to [\![A]\!]$ in the Kleisli category $\mathbb{C}_T$ for T strong

two shortcomings:

① type structure not matched by categorical constructors

② not every effectful language uses monads

<u>Power-Robinson</u>: $[\![\Gamma \vdash M : A]\!]$ is an arrow $[\![\Gamma]\!] \to [\![A]\!]$ in a premonoidal category

Premonoidal categories

given

$$\Gamma \vdash M_1 : A_1$$

$$\Delta \vdash M_2 : A_2$$

$$x_1 : A_1,\ x_2 : A_2 \vdash P : B$$

$$\begin{tikzcd}
& \llbracket \Gamma \rrbracket \times \llbracket \Delta \rrbracket \arrow[dl, "\llbracket M_1 \rrbracket \times id"'] \arrow[dr, "id \times \llbracket M_2 \rrbracket"] & \\
\llbracket A_1 \rrbracket \times \llbracket \Delta \rrbracket \arrow[dr, "id \times \llbracket M_2 \rrbracket"'] & \neq & \llbracket \Gamma \rrbracket \times \llbracket A_2 \rrbracket \arrow[dl, "\llbracket M_1 \rrbracket \times id"] \\
& \llbracket A_1 \rrbracket \times \llbracket A_2 \rrbracket \arrow[d, "\llbracket P \rrbracket"] & \\
& \llbracket B \rrbracket &
\end{tikzcd}$$

unless one M_i a value/pure

then

$$\begin{array}{l} \text{let } x_1 = M_1 \text{ in} \\ \quad \text{let } x_2 = M_2 \text{ in } P \end{array} \quad \neq \quad \begin{array}{l} \text{let } x_2 = M_2 \text{ in} \\ \quad \text{let } x_1 = M_1 \text{ in } P \end{array}$$

PREMONOIDAL CATEGORIES

given

$f : \Gamma \longrightarrow A$

$g : \Delta \longrightarrow B$

then got <u>two distinct</u> maps $\Gamma \otimes \Delta \to A \otimes B$

unless f or g is <u>central</u>

$$
\begin{tikzcd}
& \Gamma \otimes \Delta \arrow[dl, "f \otimes id"'] \arrow[dr, "id \otimes g"] & \\
A \otimes \Delta \arrow[dr, "id \otimes g"'] & \neq & \Gamma \otimes B \arrow[dl, "f \otimes id"] \\
& A \otimes B &
\end{tikzcd}
$$

PREMONOIDAL CATEGORIES

A premonoidal category $\mathbb{C}$ consists of:

- a category $\mathbb{C}$ with a unit $I \in \mathbb{C}$
- a mapping $\otimes : \mathrm{ob}\,\mathbb{C} \times \mathrm{ob}\,\mathbb{C} \longrightarrow \mathrm{ob}\,\mathbb{C}$
- for every $A, B \in \mathbb{C}$, functors $A \rtimes (-), (-) \ltimes B : \mathbb{C} \to \mathbb{C}$

  s.t. $A \rtimes B = A \otimes B = A \ltimes B$ on objects
- central natural isomorphisms α, λ, ρ + triangle, pentagon laws

PREMONOIDAL CATEGORIES

An arrow $f : A \to A'$ in $\mathbb{C}$ is <u>central</u> if

$\forall g.$
$$\begin{tikzcd}
 & A \otimes B \arrow[dl, "f \ltimes B"'] \arrow[dr, "A \rtimes g"] & \\
A' \otimes B \arrow[dr, "A' \rtimes g"'] & & A \otimes B' \arrow[dl, "f \ltimes B'"] \\
 & A' \otimes B' &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & B \otimes A \arrow[dl, "g \ltimes A"'] \arrow[dr, "B \rtimes f"] & \\
B' \otimes A \arrow[dr, "B' \rtimes f"'] & & B \otimes A' \arrow[dl, "g \ltimes A'"] \\
 & B' \otimes A' &
\end{tikzcd}$$

The wide subcategory of central maps is <u>monoidal</u>.

Premonoidal Categories

$$\left[\!\left[\begin{array}{l} \text{let } x = M_1 \text{ in} \\ \quad \text{let } x_2 = M_2 \text{ in} \\ \qquad P \end{array}\right]\!\right] =$$

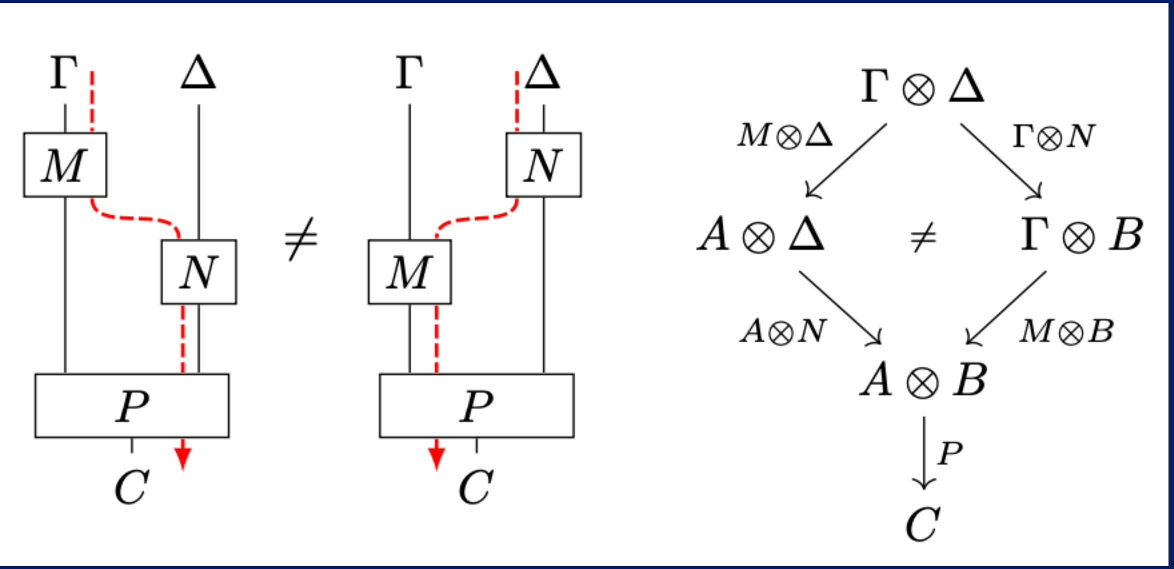
$$\begin{tikzcd}
[\![\Gamma]\!] \otimes [\![\Delta]\!] \arrow[d, "{[\![\Gamma]\!] \times [\![M_2]\!]}"] \\
[\![\Gamma]\!] \otimes [\![A_2]\!] \arrow[d, "{[\![M_1]\!] \times [\![A_2]\!]}"] \\
[\![A_1]\!] \otimes [\![A_2]\!] \arrow[d, "{[\![P]\!]}"] \\
[\![B]\!]
\end{tikzcd}$$

if this a value, could also run it first?

$\subset$ central maps are values

PREMONOIDAL CATEGORIES

describe control flow:



PREMONOIDAL CATEGORIES

eg// for (T, st) on $\mathbb{C}$, $\mathbb{C}_T$ is premonoidal:

$$A \otimes B := A \times B$$

$$A \ltimes g := \left(A \times B \xrightarrow{A \times g} A \times T(B') \xrightarrow{st} T(A \times B')\right)$$

$$f \rtimes B := \left(A \times B \xrightarrow{f \times B} T(A') \times B \underbrace{\longrightarrow}_{\substack{\text{built from} \\ st \,+\, \text{symmetry}}} T(A' \times B)\right)$$

NB: not all examples like this!

PREMONOIDAL CATEGORIES

eg//

for $\mathbb{C}, \mathbb{D}$ categories, the category $[\mathbb{C}, \mathbb{D}]_u$ of functors and $\underbrace{\text{unnatural transformations}}$ is premonoidal.

ie. families of maps

$\{FA \xrightarrow{\sigma_A} GA \mid A \in \mathbb{C}\}$

PREMONOIDAL CATEGORIES

axiomatise + organise the denotational semantics of effectful programs ...

... but some new models don't form categories, but bicategories

Bicategories

BICATEGORIES

- 2-categories with unit + associativity laws up to coherent isomorphism

- typically arise where composition is defined by a universal property

  eg// for more intensional semantic models

BICATEGORIES

eg// Spans as generalised relations:

$aRb \iff \exists x.\ \begin{matrix} fx = a \\ gx = b \end{matrix}$

maps $A \nrightarrow B$ =

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

composition by pullback =

$$\begin{tikzcd} & & R;S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\ & R \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

BICATEGORIES

eg// Spans as generalised relations :

$$
\begin{tikzcd}
 & & & (R;S);T \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\vee"] & & & \\
 & & R;S \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & & & \\
 & R \arrow[dl] \arrow[dr] & {} & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
A & & B & & C & & D
\end{tikzcd}
\neq
\begin{tikzcd}
 & & & R;(S;T) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\vee"] & & & \\
 & & & {} & S;T \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
 & R \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & {} & T \arrow[dl] \arrow[dr] & \\
A & & B & & C & & D
\end{tikzcd}
$$

BICATEGORIES

eg// Spans as generalised relations:

$$
\begin{tikzcd}[column sep=small]
 & & & (R;S);T \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\vee"] & & & \\
 & & R;S \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & & & \\
 & R \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
A & & B & & C & & D
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}[column sep=small]
 & & & R;(S;T) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\vee"] & & & \\
 & & & & S;T \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
 & R \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
A & & B & & C & & D
\end{tikzcd}
$$

BICATEGORIES

objects $A, B, \ldots$

maps / 1-cells $f, g, \ldots : A \longrightarrow B$

2-cells $\sigma, \tau, \ldots : f \Rightarrow g$

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "g"'{name=G}] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd}$$

BICATEGORIES

objects $A, B, \ldots$

maps / 1-cells $f, g, \ldots : A \to B$

2-cells $\sigma, \tau, \ldots : f \Rightarrow g$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd}$$

horizontal composition

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\tau"] & C \end{tikzcd}$$

vertical composition

$$\begin{tikzcd}[row sep=large] A \arrow[r, bend left=50, "f"] \arrow[r] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, shift left=3, "\Downarrow\sigma"] \arrow[r, phantom, shift right=3, "\Downarrow\sigma'"] & B \end{tikzcd}$$

BICATEGORIES

objects $A, B, \ldots$

maps / 1-cells $f, g, \ldots : A \longrightarrow B$

2-cells $\sigma, \tau, \ldots : f \Rightarrow g$

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \end{tikzcd}$$

$$F'FA \xrightarrow{\tau_{FA}} G'FA \xrightarrow{G'\sigma_A} G'GA$$

horizontal composition

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\tau"] & C \end{tikzcd}$$

vertical composition

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\sigma"] \arrow[r, phantom, shift right=2ex, "\Downarrow\sigma'"] & B \end{tikzcd}$$

$$\begin{tikzcd} FA \arrow[d, "\sigma_A"] \\ F'A \arrow[d, "\sigma'_A"] \\ F''A \end{tikzcd}$$

BICATEGORIES

objects $A, B, \ldots$

maps / 1-cells $f, g, \ldots : A \longrightarrow B$

2-cells $\sigma, \tau, \ldots : f \Rightarrow g$

... with horizontal composition associative + unital up to coherent isomorphism

BICATEGORIES

objects $A, B, \ldots$

maps / 1-cells $f, g, \ldots : A \longrightarrow B$

2-cells $\sigma, \tau, \ldots : f \Rightarrow g$

natural isomorphisms

$$\lambda_f : \mathrm{Id} \circ f \overset{\cong}{\Longrightarrow} f \quad , \quad \rho_f : f \circ \mathrm{Id} \overset{\cong}{\Longrightarrow} f$$

$$\alpha_{f,g,h} : (f \circ g) \circ h \overset{\cong}{\Longrightarrow} f \circ (g \circ h)$$

+ triangle and pentagon laws

BICATEGORIES

eg// $\mathrm{Span}(\mathcal{C})$ has

$$aRb \iff \exists x.\ \begin{matrix} f(x) = a \\ g(x) = b \end{matrix}$$

- objects: those of $\mathcal{C}$
- 1-cells $A \nrightarrow B$: spans $\qquad A \xleftarrow{f} R \xrightarrow{g} B$
- composition by pullback
- 2-cells as commuting maps
- α, λ, ρ from universal property //

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[dd, dashed] & \\ A & & B \\ & S \arrow[ul] \arrow[ur] & \end{tikzcd}$$

BICATEGORIES

eg// for $\mathbb{C}$ monoidal, $\mathrm{Para}(\mathbb{C})$ has: [also works with actions]

- objects: those of $\mathbb{C}$
- 1-cells $A \nrightarrow B$: pairs $P \in \mathbb{C}$, $f : P \otimes A \to B$
- 2-cells: reparameterisations

$$\begin{tikzcd} P \otimes A \arrow[r, "f"] \arrow[d, "\sigma \otimes A"'] & B \\ Q \otimes A \arrow[ur, "g"'] & \end{tikzcd}$$

- identity: $I \otimes A \xrightarrow{\cong} A$
- composition:

$$(P \otimes Q) \otimes A \xrightarrow{\cong} P \otimes (Q \otimes A) \xrightarrow{P \otimes f} P \otimes B \xrightarrow{g} C$$

//

BICATEGORIES

def : a graded monad is a monoidal functor

$$T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}].$$

monoidal category of grades

$$T_e : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\mu : T_{e'} \circ T_e \Longrightarrow T_{e' \otimes e}$$

$$\eta : \mathrm{Id} \Longrightarrow T_I$$

eg// $\mathbb{E} := (\mathbb{N}, +, 0)$

$L_n(X) = \{$ lists of length $\leq n$ over $X \}$

BICATEGORIES

def: a strong graded monad is a monoidal functor

$$T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]_{strong}$$

monoidal category of grades

$$T_e : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\mu : T_{e'} \circ T_e \Longrightarrow T_{e' \otimes e} \quad \text{strong}$$

$$\eta : \mathrm{Id} \Longrightarrow T_I \quad \text{strong}$$

BICATEGORIES

eg// for $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]_{\text{strong}}$ strong graded monad, Kl_T has:

- objects : those of $\mathbb{C}$
- maps $A \nrightarrow B$: grade e with $f : A \longrightarrow T_e B$
- 2-cells : re-gradings $\sigma : e \longrightarrow e'$
- $g \circ f := \left(A \xrightarrow{f} T_e B \xrightarrow{T_e(g)} T_e T_{e'} C \xrightarrow{\mu} T_{e \otimes e'} C \right)$

BICATEGORIES

eg// for $T : \mathbb{E} \longrightarrow [\mathbb{C}, \mathbb{C}]_{\text{strong}}$ strong graded monad,

Kl_T has: $\boxed{Kl_T := Para(T)^{op}}$

- objects : those of $\mathbb{C}$
- maps $A \nrightarrow B$: grade e with $f : A \longrightarrow T_e B$
- 2-cells : re-gradings $\sigma : e \longrightarrow e'$
- $g \circ f := \left(A \xrightarrow{f} T_e B \xrightarrow{T_e(g)} T_e T_{e'} C \xrightarrow{\mu} T_{e \otimes e'} C \right)$

in general

"bicategorify"

= replace equations by isomorphisms, $\underbrace{\text{subject to equations}}_{\text{difficult bit = which equations?}}$

in general

properties become data

"in what sense" is it true

"bicategorify"

= replace equations by isomorphisms, $\underbrace{\text{subject to equations}}$

difficult bit = which equations?

a <u>pseudofunctor</u> $F : \mathcal{B} \longrightarrow \mathcal{C}$ consists of :

- $F : \mathrm{ob}\,\mathcal{B} \longrightarrow \mathrm{ob}\,\mathcal{C}$
- for each $A, B \in \mathcal{B}$, a functor $\mathcal{B}(A,B) \longrightarrow \mathcal{C}(FA, FB)$
- coherent isos $F(\mathrm{Id}) \underset{\psi}{\cong} \mathrm{Id}$ and $F(f) \circ F(g) \underset{\phi}{\cong} F(f \circ g)$.

a <u>pseudonatural transformation</u> $\sigma : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$

consists of:

- for each $A \in \mathcal{B}$ a 1-cell $\sigma_A : FA \to GA$
- for each $f : A \to B$ in $\mathcal{B}$ a 2-cell $\bar{\sigma}_f$, compatible with Id and $\circ$

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\sigma_A"'] \arrow[dr, phantom, "\bar{\sigma}_f \; \cong"] & FB \arrow[d, "\sigma_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}.$$

"in what sense" is σ natural

a <u>modification</u> $\Gamma : \sigma \Rrightarrow \tau : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$

consists of :

- a 2-cell $\begin{tikzcd} FA \arrow[r, bend left, "\sigma_A"] \arrow[r, bend right, "\tau_A"'] \arrow[r, phantom, "\Downarrow \Gamma_A"] & GA \end{tikzcd}$ for each $A \in \mathcal{B}$

compatible with $\bar{\sigma}_f$ and $\bar{\tau}_f$

EXAMPLE: MONOIDAL BICATEGORIES

EXAMPLE: MONOIDAL BICATEGORIES

A monoidal category $(\mathbb{C}, \otimes, I)$ has:

- $I \in \mathbb{C}$
- $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$
- natural isos α, λ, ρ

such that Δ + ⬡ hold

EXAMPLE: MONOIDAL BICATEGORIES

A monoidal bicategory $(\mathcal{B}, \otimes, I)$ has:

- $I \in \mathcal{B}$
- $\otimes : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$
- pseudo natural equivalences α, λ, ρ
- modifications filling $\triangle$ + ⬠

... subject to equations

Example : monoidal bicategories

$$\begin{tikzcd}
((AB)C)D \arrow[r, "\alpha"] \arrow[d, "\alpha D"'] & (AB)(CD) \arrow[r, "\alpha"] \arrow[d, phantom, "\mathfrak{p}\Uparrow"] & A(B(CD)) \\
(A(BC))D \arrow[rr, "\alpha"'] & {} & A((BC)D) \arrow[u, "A\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
& AB & \\
(AI)B \arrow[ur, "\rho B"] \arrow[rr, "\alpha"'] & {} \arrow[u, phantom, "\mathfrak{m}\Uparrow"] & A(IB) \arrow[ul, "A\lambda"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(IA)B \arrow[r, "\alpha"] \arrow[d, "\lambda B"'] \arrow[dr, phantom, "\overset{\mathfrak{l}}{\Rightarrow}"] & I(AB) \arrow[dl, bend left, "\lambda"] \\
AB &
\end{tikzcd}
\qquad
\begin{tikzcd}
(AB)I \arrow[r, "\alpha"] \arrow[dr, bend right, "\rho"'] & A(BI) \arrow[d, "A\rho"] \arrow[dl, phantom, "\overset{\mathfrak{r}}{\Rightarrow}"] \\
& AB
\end{tikzcd}$$

Example: Para($\mathcal{C}$) is monoidal

Recall:

$$
\begin{tikzcd} A \arrow[r, "+" marking] & B \end{tikzcd} = \left(P,\ P \otimes A \xrightarrow{f} B \right)
$$

$$
\begin{tikzcd}
A \arrow[r, bend left, "+" marking, ""{name=U, below}] \arrow[r, bend right, "+" marking, ""{name=D, above}] & B
\arrow[Rightarrow, from=U, to=D]
\end{tikzcd}
=
\left(
\begin{tikzcd}[row sep=small]
P \otimes A \arrow[d] \arrow[dr] & \\
& B \\
Q \otimes A \arrow[ur] &
\end{tikzcd}
\right)
$$

EXAMPLE : Para($\mathbb{C}$) IS MONOIDAL

- Every $f : A \to B$ in $\mathbb{C}$ defines $\tilde{f} := (I \otimes A \xrightarrow{\cong} A \xrightarrow{f} B)$ in Para($\mathbb{C}$)
- If f is an iso, $\tilde{f}$ is an equivalence

EXAMPLE: Para($\mathbb{C}$) IS MONOIDAL

- Every $f : A \to B$ in $\mathbb{C}$ defines $\tilde{f} := \left(I \otimes A \xrightarrow{\cong} A \xrightarrow{f} B \right)$ in Para($\mathbb{C}$)
- If f is an iso, $\tilde{f}$ is an equivalence

So we get:

- unit I ; $A \tilde{\otimes} B := A \otimes B$
- $\tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}$ give equivalences $A \otimes I \xrightarrow{\cong} A$ etc
- modifications P, M, L, R from coherence

DEFINING PREMONOIDAL BICATEGORIES

"bicategorify" = replace equations by isomorphisms, subject to equations

properties become data

A premonoidal category $\mathbb{C}$ consists of:

- a category $\mathbb{C}$ with a unit $I \in \mathbb{C}$
- a mapping $\otimes : \mathrm{ob}\,\mathbb{C} \times \mathrm{ob}\,\mathbb{C} \longrightarrow \mathrm{ob}\,\mathbb{C}$
- for every $A, B \in \mathbb{C}$, functors $A \rtimes (-), (-) \ltimes B : \mathbb{C} \to \mathbb{C}$

  s.t. $A \rtimes B = A \otimes B = A \ltimes B$ on objects
- central natural isomorphisms α, λ, ρ

  subject to triangle + pentagon laws

A **premonoidal bicategory** $\mathcal{B}$ consists of :

- a bicategory $\mathcal{B}$ with a <u>unit</u> $I \in \mathcal{B}$
- a mapping $\otimes : \mathrm{ob}\,\mathcal{B} \times \mathrm{ob}\,\mathcal{B} \longrightarrow \mathrm{ob}\,\mathcal{B}$
- for every $A, B \in \mathcal{C}$, pseudo functors $A \rtimes (-), (-) \ltimes B : \mathcal{B} \to \mathcal{B}$
  s.t. $A \rtimes B = A \otimes B = A \ltimes B$ on objects

⚠ • <u>central</u> pseudo natural equivalences α, λ, ρ

⚠ • invertible modifications witnessing axioms + equations of a monoidal bicategory

CENTRALITY IN PREMONOIDAL CATS

An arrow $f : A \to A'$ in $\mathbb{C}$ is <u>central</u> if

$$\forall g. \quad \begin{tikzcd}[column sep=small] & A \otimes B \arrow[dl, "f \ltimes B"'] \arrow[dr, "A \rtimes g"] & \\ A' \otimes B \arrow[dr, "A' \rtimes g"'] & & A \otimes B' \arrow[dl, "f \ltimes B'"] \\ & A' \otimes B' & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & B \otimes A \arrow[dl, "g \ltimes A"'] \arrow[dr, "B \rtimes f"] & \\ B' \otimes A \arrow[dr, "B' \rtimes f"'] & & B \otimes A' \arrow[dl, "g \ltimes A'"] \\ & B' \otimes A' & \end{tikzcd}$$

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\forall g.\quad
\begin{tikzcd}
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A' \otimes B \arrow[dr, "A' \rtimes g"'] & & A \otimes B' \arrow[dl, "f \ltimes B'"] \\
 & A' \otimes B' &
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
 & B \otimes A \arrow[dl, "g \ltimes A"'] \arrow[dr, "B \rtimes f"] & \\
B' \otimes A \arrow[dr, "B' \rtimes f"'] & & B \otimes A' \arrow[dl, "g \ltimes A'"] \\
 & B' \otimes A' &
\end{tikzcd}
$$

$$\left(A \rtimes B = A \ltimes B \xrightarrow{f \ltimes B} A' \ltimes B = A' \rtimes B\right) : A \rtimes (-) \Rightarrow A' \rtimes (-)$$

$$\left(B \ltimes A = B \rtimes A \xrightarrow{B \rtimes f} B \rtimes A' = B \ltimes A'\right) : (-) \ltimes A \Rightarrow (-) \ltimes A'$$

CENTRALITY IN PREMONOIDAL CATS

An arrow $f : A \to A'$ in $\mathbb{C}$ is <u>central</u> iff

1) $\left(A \rtimes B = A \ltimes B \xrightarrow{f \ltimes B} A' \ltimes B = A' \rtimes B \right) : A \rtimes (-) \Rightarrow A' \rtimes (-)$

2) $\left(B \ltimes A = B \rtimes A \xrightarrow{B \rtimes f} B \rtimes A' = B \ltimes A' \right) : (-) \ltimes A \Rightarrow (-) \ltimes A'$

CENTRALITY IN PREMONOIDAL BICATS

A **central 1-cell** $A \to A'$ consists of

- a 1-cell $f : A \to A'$
- for every $g : B \to B'$, invertible 2-cells

$$
\begin{tikzcd}
AB \arrow[r, "A \rtimes g"] \arrow[d, "f \ltimes B"'] \arrow[dr, phantom, "lc^f_g"] & AB' \arrow[d, "f \ltimes B'"] \\
A'B \arrow[r, "A' \rtimes g"'] & A'B'
\end{tikzcd}
\qquad
\begin{tikzcd}
BA \arrow[r, "g \ltimes A"] \arrow[d, "B \rtimes f"'] \arrow[dr, phantom, "rc^f_g"] & B'A \arrow[d, "B' \rtimes f"] \\
BA' \arrow[r, "g \ltimes A'"'] & B'A'
\end{tikzcd}
$$

giving $(A \rtimes - \Rightarrow A' \rtimes -)$ and $(- \ltimes A \Rightarrow - \ltimes A')$.

CENTRALITY IN PREMONOIDAL BICATS

A <u>central 2-cell</u> $(f, lc^f, rc^f) \Rightarrow (f', lc^{f'}, rc^{f'})$

is a 2-cell $\sigma : f \Rightarrow f'$ giving modifications

$$\begin{tikzcd} AB \arrow[r, bend left, "f \ltimes B"] \arrow[r, bend right, "f' \ltimes B"'] \arrow[r, phantom, "\Downarrow \sigma \ltimes B"] & A'B \end{tikzcd} \qquad \begin{tikzcd} BA \arrow[r, bend left, "B \rtimes f"] \arrow[r, bend right, "B \rtimes f'"'] \arrow[r, phantom, "B \rtimes \sigma \Downarrow"] & BA' \end{tikzcd}$$

between the associated pseudonatural trans.

EXAMPLES

eg: • any pseudomonad on $\mathcal{C}at$
• strong monad on $\mathbb{C}$ $\Rightarrow$ strong pseudomonad on $\mathrm{Para}(\mathbb{C})$, $\mathrm{Span}(\mathbb{C})$
• $(-)\otimes M$ for M a pseudomonoid
• any pseudomonad WRT $(0,+)$

- if T is a strong pseudomonad on $\mathcal{B}$, $\mathcal{B}_T$ is premonoidal
- if T is a strong graded monad on $\mathcal{B}$, Kl_T is premonoidal
- $[\mathcal{B},\mathcal{C}]_u$, bicategory of pseudofunctors + unnatural transform.

Some subtleties

- centrality as data ⇒ not clear centre is monoidal
  - in general $\mathrm{lc}^f_g \neq (\mathrm{rc}^g_f)^{-1}$
- modifications in definition require some care
  - have both $(\lambda, \bar{\lambda})$ and $(\lambda, \mathrm{lc}^{\lambda})$

$$
\begin{tikzcd}
& (-B)(CD) \arrow[dr, "\alpha_{-,B,CD}"] & \\
((-B)C)D \arrow[ur, "\alpha_{-B,C,D}"] \arrow[d, "\alpha_{-,B,C}D"'] & & (-)(B(CD)) \\
(-(BC))D \arrow[rr, "\alpha_{-,BC,D}"'] & \arrow[u, phantom, "\Uparrow \mathfrak{p}_{-,B,C,D}"] & (-)((BC)D) \arrow[u, "\mathsf{rc}^{\alpha}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& (AB)(C-) \arrow[dr, "\alpha_{A,B,C-}"] & \\
((AB)C)(-) \arrow[ur, "\alpha_{AB,C,-}"] \arrow[d, "\mathsf{lc}^{\alpha}"'] & & A(B(C-)) \\
(A(BC))(-) \arrow[rr, "\alpha_{A,BC,-}"'] & \arrow[u, phantom, "\Uparrow \mathfrak{p}_{A,B,C,-}"] & A((BC)(-)) \arrow[u, "A\alpha_{B,C,-}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (-)B & \\
(-I)B \arrow[ur, bend left, "\rho B"] \arrow[rr, "\alpha_{-,I,B}"'] & \arrow[u, phantom, "\Uparrow \mathfrak{m}_{-,B}"] & (-)(IB) \arrow[ul, bend right, "\mathsf{rc}^{\lambda}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A(-) & \\
(AI)(-) \arrow[ur, bend left, "\mathsf{lc}^{\rho}"] \arrow[rr, "\alpha_{A,I,-}"'] & \arrow[u, phantom, "\Uparrow \mathfrak{m}_{A,-}"] & A(I-) \arrow[ul, bend right, "A\lambda"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(IA)(-) \arrow[r, "\alpha_{I,A,-}"] \arrow[d, "\mathsf{lc}^{\lambda}"'] \arrow[dr, phantom, "\Rightarrow \mathfrak{l}_{A,-}"] & I(A-) \arrow[dl, bend left, "\lambda"] \\
A(-) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(-B)I \arrow[r, "\alpha_{-,B,I}"] \arrow[dr, bend right, "\rho_{-B}"'] \arrow[dr, phantom, "\Rightarrow \mathfrak{r}_{-,B}"] & (-)(BI) \arrow[d, "\mathsf{rc}^{\rho}"] \\
& AB
\end{tikzcd}
$$

WHY PREMONOIDAL BICATEGORIES

premonoidal categories

axiomatises many semantic models

+

framework for 2-dimensional semantics

bicategorical models

more refined / intensional information

eg// spans, games, Prof-based models, graded monads