Strong pseudomonads and premonoidal bicategories

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Abstract

Strong monads and premonoidal categories play a central role in clarifying the denotational semantics of effectful programming languages. Unfortunately, this theory excludes many modern semantic models in which the associativity and unit laws only hold up to coherent isomorphism: for instance, because composition is defined using a universal property.

This paper remedies the situation. We define premonoidal bicategories and a notion of strength for pseudomonads, and show that the Kleisli bicategory of a strong pseudomonad is premonoidal. As often in 2-dimensional category theory, the main difficulty is to find the correct coherence axioms on 2-cells. We therefore justify our definitions with numerous examples and by proving a correspondence theorem between actions and strengths, generalizing a well-known category-theoretic result.

1 Introduction

Moggi famously observed that the category-theoretic notion of *strong monad* encapsulates the structure of effectful programs, and hence one can give a denotational model of an effectful programming language using a strong monad on a cartesian (or more generally, monoidal) category ([63, 64]). The approach was refined by Power & Robinson, whose *premonoidal categories* axiomatize the core structure of Moggi's models [73]. These insights give a framework for studying existing models and constructing new ones, abstracting away from the particular choice of effect.

On the other hand, in recent years a plethora of models have been proposed which are not categories but *bicategories* [2]. These models come with more structure, and typically provide finer-grained or more intensional information that traditional categorical ones. Examples include various kinds of game semantics ([8, 60]), recent models of linear logic based on profunctors ([18, 19, 27]), and models describing the $\beta\eta$-rewrites of the simply-typed λ-calculus ([36, 21]).

This paper is the first step in extending the Moggi–Power–Robinson framework to these new examples. We introduce bicategorical versions of strong monads and premonoidal categories, and validate these definitions through examples and theorems paralleling the categorical theory.

We begin with a brief overview of monadic semantics (Section 1.1). Then we introduce and motivate bicategories (Section 1.2), and outline our main contributions (Section 1.3).

1.1 Strong monads and premonoidal categories

A *strong monad* on a monoidal category $(\mathbb{C}, \otimes, I)$ is a monad (T, μ, η) equipped with a natural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$ known as a *strength*, compatible with both the monoidal structure of $\mathbb{C}$ and the monadic structure of T ([46], or Section 3). In this setting, an effectful program $(\Gamma \vdash M : A)$ is modelled by a Kleisli arrow $\Gamma \to TA$ in $\mathbb{C}$ and substitution into another program $(\Delta, x : A \vdash N : B)$ is modelled using the strength and the Kleisli extension operation:

$$\Delta \otimes \Gamma \xrightarrow{\Delta \otimes M} \Delta \otimes TA \xrightarrow{t} T(\Delta \otimes A) \xrightarrow{\gg\!= N} TB.$$

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This approach provides a sound and complete class of semantic models for certain effectful languages [64]. But it also involves some indirect aspects: programs are not interpreted in $\mathbb{C}$ but in the Kleisli category $\mathbb{C}_T$. Premonoidal categories [73] remedy this by directly axiomatising the structure of $\mathbb{C}_T$ required for the semantic interpretation; they more directly reflect the structure of effectful languages in which a monad is not explicit (*e.g.* [63, 25]).

A *premonoidal category* is a category $\mathbb{D}$ equipped with a tensor product $\otimes$ that is only functorial in each argument separately. Effectful programs are interpreted directly as morphisms in $\mathbb{D}$, and the substitution above is interpreted as

$$\Delta \otimes \Gamma \xrightarrow{\Delta \otimes M} \Delta \otimes A \xrightarrow{N} B.$$

This approach strictly generalises the previous one: if $(\mathbb{C}, \otimes, I)$ is symmetric monoidal, then any strength for the monad T induces a premonoidal structure on $\mathbb{C}_T$, but there are also examples of premonoidal categories not arising from a monad (*e.g.* [83, 72, 80]).

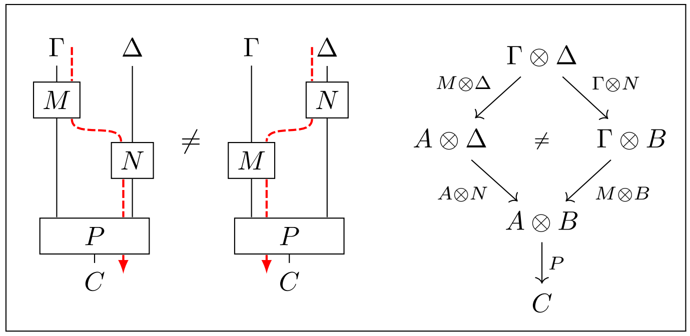


Figure 1: Control flow and data flow in a premonoidal bicategory. Here M and N represent program fragments which cannot be re-ordered—for instance, different print statements—and the dashed red line indicates control flow.

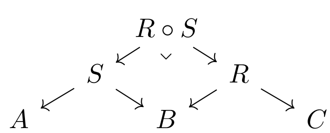
The lack of functoriality of $\otimes$ reflects the fact that one cannot generally re-order the statements of an effectful program, even if the data flow permits it. So the composition of morphisms in a premonoidal category should be understood as encoding control flow. This is illustrated in Figure 1 using the graphical calculus for premonoidal categories ([40, 76]).

1.2 The case for bicategories

As indicated above, many recent models form bicategories instead of categories. Roughly speaking, a bicategory is like a category except the unit and associativity laws for the composition of morphisms are replaced by well-behaved isomorphisms. This kind of structure typically arises when composition is defined using a universal construction; for an illustrative example we consider a bicategory of spans, that occurs widely in models of programming languages and computational processes (*e.g.* [60, 17, 31, 1]).

Example. Spans of sets

Consider a model in which objects are sets and a morphism from A to B consists of a set S and a span of functions $A \longleftarrow S \longrightarrow B$. We can compose pairs of morphisms $A \longleftarrow S \longrightarrow B$ and $B \longleftarrow R \longrightarrow C$ using a pullback in the category of sets:

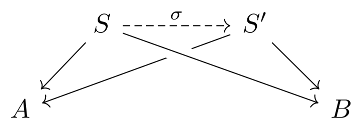


This composition correctly captures a notion of ‘plugging together’ spans, but is only associative in a weak sense. Indeed, the two ways of taking pullbacks

 (1)

are generally not equal, but they are canonically isomorphic via the universal property of pullbacks. Similarly, the span $A \xleftarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} A$ is only a weak identity for composition, because pulling back along id only gives an isomorphic set.

To describe composition in this model, therefore, we require a notion of morphism between spans. If S and S' are spans from A to B, then a map between them is a function $\sigma : S \to S'$ that commutes with the span legs on each side:



The two iterated composites in (1) are isomorphic as spans, so composition of spans is associative up to isomorphism. Similarly, the identity span is unital up to isomorphism and, because these isomorphisms arise from a universal property, they behave well together. Bicategories axiomatize such situations.

Definition 1 ([2])**.** *A bicategory $\mathscr{B}$ consists of:*

- *A collection of objects $A, B, \ldots$*
- *For all objects A and B, a collection of morphisms from A to B, themselves related by morphisms: thus we have a* hom-category *$\mathscr{B}(A, B)$ whose objects (typically denoted $f, g : A \to B$) are called* 1-cells*, and whose morphisms (typically denoted $\sigma, \tau : f \Rightarrow g$) are called* 2-cells*. The category structure means we can compose 2-cells between parallel 1-cells.*
- *For all objects A, B, and C, a composition functor $\circ_{A,B,C} : \mathscr{B}(B, C) \times \mathscr{B}(A, B) \longrightarrow \mathscr{B}(A, C)$ and, for all A, an identity 1-cell $\mathsf{Id}_A \in \mathscr{B}(A, A)$.*
- *Coherent structural 2-cells: since the composition of 1-cells is weak, we have a natural family of invertible 2-cells $\mathsf{a}_{f,g,h} : (f \circ g) \circ h \Longrightarrow f \circ (g \circ h)$ instead of the usual associativity equation. Similarly, we have natural families of invertible 2-cells $\mathsf{l}_f : \mathsf{Id}_B \circ f \Longrightarrow f$ and $\mathsf{r}_f : f \circ \mathsf{Id}_A \Longrightarrow f$ instead of the left and right identity laws. These structural 2-cells must satisfy coherence axioms similar to those for a monoidal category.*

The 2-dimensional structure of a bicategory lets one describe relationships between morphisms, which can be used to provide refined semantic information (*e.g.* [36, 21, 84, 65, 44]). To illustrate further, we consider the **Para** construction, which is a general way to build models of parametrized processes [23] (see also [6, 14, 77]). In this bicategory, the 2-cells describe how processes can be reparametrized; thus the weakness arises because we are tracking extra information. We will use this bicategory several times, so we spell out the definition in detail.

Example**: the Para construction.** Starting from a monoidal category $(\mathbb{C}, \otimes, I)$, we construct a bicategory $\mathbf{Para}(\mathbb{C})$ as follows:

- The objects are those of $\mathbb{C}$.
- A 1-cell from A to B is a parametrized $\mathbb{C}$-morphism, defined as an object $P \in \mathbb{C}$ together with a morphism $f : P \otimes A \to B$ in $\mathbb{C}$. The object P is thought of as a space of parameters.
- A 2-cell from $f : P \otimes A \to B$ to $g : P' \otimes A \to B$ is a reparametrization map, *i.e.* a map $\sigma : P \to P'$ such that $g \circ (\sigma \otimes A) = f$.

Composition of 1-cells is defined using the tensor product of parameters: if $f : P \otimes A \to B$ and $g : Q \otimes B \to C$, then $g \circ f$ is the object $Q \otimes P$ equipped with the map

$$(Q \otimes P) \otimes A \xrightarrow{\cong} Q \otimes (P \otimes A) \xrightarrow{Q \otimes f} Q \otimes B \xrightarrow{g} C$$

where the first map is the associativity of the tensor product.

If we also have $h : R \otimes C \to D$, then the two composites $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have parameter spaces $(R \otimes Q) \otimes P$ and $R \otimes (Q \otimes P)$, respectively. Because the tensor product in a monoidal category is generally only associative up to isomorphism, these 1-cells are only isomorphic in $\mathbf{Para}(\mathbb{C})$. A similar argument applies to the identity laws, so $\mathbf{Para}(\mathbb{C})$ is a bicategory with a, l and r given by $\mathbb{C}$'s monoidal structure. In fact, if $\mathbb{C}$ is symmetric monoidal, this lifts to a symmetric monoidal structure on $\mathbf{Para}(\mathbb{C})$: see Section 2.3.

1.3 This paper: semantics for effects in bicategories

The central aim of this paper is to 'bicategorify' the definitions of strong monad and premonoidal category. A definition in category theory typically involves data subject to equations expressed as commutative diagrams. In bicategory theory these equations are replaced by invertible 2-cells filling each diagram, and then we have equations on these 2-cells. The main challenge, therefore, is knowing what the 2-cell equations should be. This can be difficult and subtle, even for experts. For example, symmetric monoidal structure is apparently innocuous and well-understood, but the history of its bicategorical definition is littered with missteps: see [78, §2.1] for an overview. We therefore stipulate the following necessary (but not necessarily sufficient) desiderata for the 'correctness" of a bicategorical definition:

1. It's not too strict to capture the relevant examples;
2. It's not too weak to prove results one would expect, so the 1-dimensional theory lifts to the 2-dimensional setting;
3. Some form of coherence theorem holds.

In this paper we tackle (1) and (2): we justify our definitions both by providing examples and by showing that our definitions are consistent with other standard bicategorical constructions. This follows a pattern common in the literature, where the difficulty of coherence proofs means that one typically wants to fix a definition before proving coherence (see, for instance, the long gap between the introduction of pseudomonoids [42] and the first coherence proof we are aware of [47]).[1]

Contributions. First, we define a notion of *strength* for pseudomonads (Section 3) and give a range of examples, including a bicategorical version of the fact that every monad on **Set** is canonically strong (Section 3.3).

Next we show that, just as strengths on a monad correspond to certain actions on its Kleisli category (see *e.g.* [57, Proposition 4.3]), so strengths for a pseudomonad correspond to certain actions on its Kleisli bicategory (Section 4 and Theorem 1). This correspondence is well-known in the categorical setting, where actions are sometimes used directly to axiomatise models for effectful languages (see *e.g.* [62]).

In Section 6 we turn to premonoidal structure. We introduce premonoidal bicategories and show that the Kleisli bicategory of a strong pseudomonad is premonoidal (Theorem 2). We also observe that *graded* monads ([79, 59, 43]) which track quantitative information about side effects, have a natural notion of Kleisli bicategory and that this is premonoidal.

We finish with a summary of the main results and related work, and suggestions for further investigation (Section 7).

2 Background: pseudofunctors, pseudomonads, and monoidal bicategories

Many concepts in category theory have corresponding versions for bicategories. We first summarise the basic definitions of pseudofunctors, pseudonatural transformations, and modifications (Section 2.1), then discuss the bicategorical notions of monad (Section 2.2) and monoidal structure (Section 2.3) we need for this paper. For reasons of space we only give a brief outline and omit the coherence axioms. For a full overview of the basic bicategorical definitions, see [50]; for the definition of (symmetric) monoidal bicategories, including many beautiful diagrams, see [81]. Readable introductions to the wider subject of bicategories include [2, 41]; a more theoretical-computer science perspective is available in [70, 71].

[1] It turns out that our definition of strong pseudomonad *is* coherent. However, like many coherence results the proof is both long and technical, so we shall present it elsewhere.

2.1 Basic notions

Morphisms of bicategories are called pseudofunctors. Just as bicategories are categories 'up to isomorphism', so pseudofunctors are functors 'up to isomorphism'.

Definition 2. *A* pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ *consists of:*

- *A mapping* $F : ob(\mathscr{B}) \to ob(\mathscr{C})$ *on objects;*
- *A functor* $F_{A,B} : \mathscr{B}(A, B) \to \mathscr{C}(FA, FB)$ *for every* $A, B \in \mathscr{B}$*;*
- *A* unitor $\psi_A : \mathsf{Id}_{FA} \overset{\cong}{\Longrightarrow} F(\mathsf{Id}_A)$ *for every* $A \in \mathscr{B}$*;*
- *A* compositor $\phi_{f,g} : F(f) \circ F(g) \overset{\cong}{\Longrightarrow} F(f \circ g)$ *for every composable pair of 1-cells* f *and* g*, natural in* f *and* g*.*

This data is subject to three axioms similar to those for strong monoidal functors (see e.g. *[50]).*

We generally abuse notation by referring to a pseudofunctor (F, ϕ, ψ) simply as F; where there is no risk of confusion, we shall employ similar abuses for structure throughout. A pseudofunctor is called *strict* if ϕ and ψ are both the identity.

Example 1. *Every endofunctor* F *on a monoidal category* $(\mathbb{C}, \otimes, I)$ *with a strength* $t_{A,B} : A \otimes F(B) \to F(A \otimes B)$ *(see* e.g. *[46]) determines a strict endo-pseudofunctor* $\widetilde{F}$ *on* $\mathbf{Para}(\mathbb{C})$*. The action on objects is the same, and on 1-cells* $\widetilde{F}(P \otimes A \xrightarrow{f} B)$ *is the object* P *together with the composite* $\big(P \otimes FA \xrightarrow{t} F(P \otimes A) \xrightarrow{Ff} FB\big)$*.*

Transformations between pseudofunctors are like natural transformations, except one must say in what sense naturality holds for each 1-cell.

Definition 3. *For pseudofunctors* $F, G : \mathscr{B} \to \mathscr{C}$*, a* pseudonatural transformation $\eta : F \Rightarrow G$ *consists of:*

- *A 1-cell* $\eta_A : FA \to GA$ *for every* $A \in \mathscr{B}$*;*
- *For every* $f : A \to B$ *in* $\mathscr{B}$ *an invertible 2-cell*

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & \overset{\overline{\eta}_f}{\Leftarrow} & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array} \tag{2}$$

natural in f *and satisfying identity and composition laws.*

Example 2. *Every natural transformation* $\sigma : F \Rightarrow F'$ *between strong endofunctors* (F, s) *and* (G, t) *which is compatible with the strengths ('strong natural transformation': see* e.g. *[57]) determines a pseudonatural transformation* $\widetilde{\sigma} : \widetilde{F} \Rightarrow \widetilde{G}$ *on* $\mathbf{Para}(\mathbb{C})$*. Each component* $(\widetilde{\sigma})_A$ *is just* $\widetilde{\sigma_A}$*, and for a 1-cell* $f : P \otimes A \to B$ *the 2-cell* $\overline{\widetilde{\sigma}}_f$ *witnessing naturality is the canonical isomorphism* $I \otimes P \xrightarrow{\cong} P \otimes I$ *in* $\mathbb{C}$*.*

Because bicategories have a second layer of structure, there is also a notion of map between pseudonatural transformations.

Definition 4. *A* modification $\boldsymbol{m} : \eta \to \theta$ *between pseudonatural transformations* $F \Rightarrow G : \mathscr{B} \to \mathscr{C}$ *consists of a 2-cell* $\boldsymbol{m}_B : \eta_B \Rightarrow \theta_B$ *for every* $B \in \mathscr{B}$*, subject to an axiom expressing compatibility between* $\boldsymbol{m}$ *and each* $\overline{\eta}_f$ *and* $\overline{\theta}_f$*.*

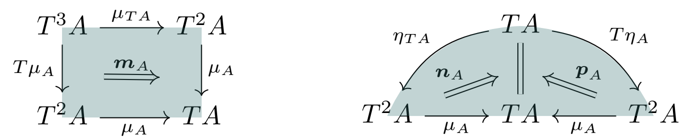
For any bicategories $\mathscr{B}$ and $\mathscr{C}$ there exists a bicategory $\mathrm{Hom}(\mathscr{B}, \mathscr{C})$ with objects pseudofunctors, 1-cells pseudonatural transformations, and 2-cells modifications. There is also a product bicategory $\mathscr{B} \times \mathscr{C}$ formed component-wise: objects are pairs (B, C), and 1-cells and 2-cells are obtained using the cartesian product of categories: $(\mathscr{B} \times \mathscr{C})((B, C), (B', C')) := \mathscr{B}(B, B') \times \mathscr{C}(C, C')$.

2.2 Pseudomonads and Kleisli bicategories

The bicategorical correlate of a monad is a *pseudomonad*.

Definition 5 ([54]). *A* pseudomonad *on a bicategory $\mathscr{B}$ consists of a pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ equipped with:*

- Unit *and* multiplication *pseudonatural transformations $\eta : \mathrm{id} \Rightarrow T$ and $\mu : T^2 \Longrightarrow T$, where $T^2 = T \circ T$;*
- *Invertible modifications $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}$ with components*



replacing the usual monad laws, and satisfying two further coherence axioms (see e.g. *[47]).*

Pseudomonads have a rich theory (*e.g.* [54, 55, 13]). Here we explore their application to denotational semantics but there are many other applications, including universal algebra (in the study of variable binding [37, 20] and operads [30]) and formal category theory (*e.g.* [68]). The dual notion of pseudo*comonad* also arises naturally, for instance in the context of linear logic (*e.g.* [19, 66, 27]).

A simple example is given by the Writer pseudomonad on **Cat**, the bicategory with objects small categories, 1-cells functors, and 2-cells natural transformations. The structural isomorphisms a, l and r are all the identity (giving a *2-category*).

Example 3. *Let $(\mathbb{C}, \otimes, I)$ be a monoidal category. The pseudofunctor $(-) \times \mathbb{C} : \mathbf{Cat} \to \mathbf{Cat}$ has a pseudomonad structure with 1-cell components*

$$\begin{aligned} \eta_{\mathbb{D}} &= \mathbb{D} \xrightarrow{\cong} \mathbb{D} \times 1 \xrightarrow{\mathbb{D} \times I} \mathbb{D} \times \mathbb{C} \\ \mu_{\mathbb{D}} &= (\mathbb{D} \times \mathbb{C}) \times \mathbb{C} \xrightarrow{\cong} \mathbb{D} \times (\mathbb{C} \times \mathbb{C}) \xrightarrow{\mathbb{D} \times \otimes} \mathbb{D} \times \mathbb{C} \end{aligned}$$

and 2-cell components $\boldsymbol{m}, \boldsymbol{n}$ and $\boldsymbol{p}$ given by the associator and unitors for the monoidal structure in $\mathbb{C}$.

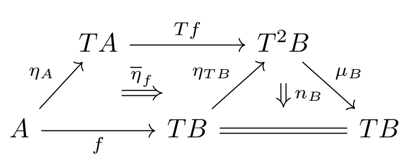
Example 4. *Every strong monad (T, μ, η, t) on a monoidal category $(\mathbb{C}, \otimes, I)$ determines a pseudomonad on $\mathbf{Para}(\mathbb{C})$: the underlying pseudofunctor is $\widetilde{T}$ and the pseudonatural transformations are $\widetilde{\mu}$ and $\widetilde{\eta}$. (This remains true if the monoidal structure is replaced by an* action *[67].)*

Just as monads have Kleisli categories, pseudomonads have Kleisli bicategories [10]. If T is a pseudomonad on $\mathscr{B}$, the *Kleisli bicategory $\mathscr{B}_T$* has the same objects as $\mathscr{B}$ and hom-categories $\mathscr{B}_T(A, B) := \mathscr{B}(A, TB)$. The identity on A is the 1-cell $\eta_A \in \mathscr{B}(A, TA)$ and the composite of $f \in \mathscr{B}(A, TB)$ and $g \in \mathscr{B}(B, TC)$ is defined to be

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC. \tag{3}$$

The structural 2-cells $\mathsf{a}, \mathsf{l}, \mathsf{r}$ in $\mathscr{B}_T$ are constructed using the pseudomonad structure.

For example, for $f \in \mathscr{B}_T(A, B)$, precomposing with η_A satisfies the identity law up to the invertible 2-cell



which becomes the component r_f.

In (3), and in the "pasting" diagram above, we have drawn the composition of morphisms in the bicategory $\mathscr{B}$ as we would in a category, ignoring issues of weak associativity and identity. This relies on *coherence*: see Section 2.4 for a discussion.

We finish with a final class of examples arising from universal algebra. These are some of the key instances of pseudomonads, but less important for our concerns here.

Example 5. *Just as algebraic structure on a set is described by monads on* **Set***, so many examples of algebraic structure on categories can be described by pseudomonads on* **Cat***. Examples include completions under a class of colimits, or adding finite products: see* e.g. *[10, 38].*

Similarly, many structures of interest in the study of substitution and variable binding arise as internal monads (see [82]) in the Kleisli bicategory of a 2-monad on **Cat** *which is extended canonically to a pseudomonad on* **Prof** *via the theory of [20]. For example, by taking the free finite product 2-monad on* **Cat***, one may identify many-sorted algebraic theories (clones) as internal monads in the associated Kleisli bicategory over* **Prof** *[37].*

2.3 Monoidal bicategories

A monoidal bicategory is a bicategory equipped with a unit object and a tensor product which is only weakly associative and unital. To motivate the various components of the construction, we explain how a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ induces a monoidal structure on $\mathbf{Para}(\mathbb{C})$, with the same action on objects.

The idea is that we can combine the parameters using $\otimes$. For 1-cells $f : P \otimes A \to B$ and $g : P' \otimes A' \to B'$, we set $f \mathbin{\widetilde{\otimes}} g$ to be the object $P \otimes P'$ equipped with

$$(P \otimes P') \otimes (A \otimes A') \xrightarrow{\cong} (P \otimes A) \otimes (P' \otimes A') \xrightarrow{f \otimes g} B \otimes B'$$

where the first map is defined using the symmetry of $\otimes$. On 2-cells, we use the tensor product of maps in $\mathbb{C}$. This construction does not strictly preserve identities and composition, but it does preserve them up to isomorphism. Thus, we get a pseudofunctor $\widetilde{\otimes} : \mathbf{Para}(\mathbb{C}) \times \mathbf{Para}(\mathbb{C}) \longrightarrow \mathbf{Para}(\mathbb{C})$.

We examine the sense in which this tensor is associative and unital, by lifting the structural isomorphisms from $\mathbb{C}$. Note that every map $f : A \to B$ in $\mathbb{C}$ determines a 1-cell $\widetilde{f}$ in $\mathbf{Para}(\mathbb{C})$ given by the object I and the composite $(I \otimes A \xrightarrow{\cong} A \xrightarrow{f} B)$, where $\cong$ is the unit isomorphism. If f has an inverse f^{-1}, the composite $\widetilde{f} \circ \widetilde{f^{-1}}$ has parameter $I \otimes I$ and thus cannot be the identity. But it is isomorphic to the identity: the pair $(\widetilde{f}, \widetilde{f^{-1}})$ is known as an *equivalence* (an 'isomorphism up to isomorphism'). Thus, although the tensor $\otimes$ on $\mathbb{C}$ is associative and unital up to isomorphism, the tensor $\widetilde{\otimes}$ on $\mathbf{Para}(\mathbb{C})$ is only associative and unital up to equivalence. The structural 1-cells are all pseudonatural in a canonical way (Example 2).

Following the general pattern of bicategorification, the triangle and pentagon axioms of a monoidal category now only hold up to isomorphism: one route round the pentagon has three sides and the other has two, so one composite has parameter $I^{\otimes 3}$ and the other has parameter $I^{\otimes 2}$. These are canonically isomorphic, so we get families of invertible 2-cells witnessing the categorical axioms. All the structure we have defined so far has used the canonical isomorphisms of $\mathbb{C}$, so these families are actually modifications on $\mathbf{Para}(\mathbb{C})$. Moreover, by the axioms of a monoidal category, these structural modifications satisfy axioms of their own.

In summary, a monoidal bicategory is a bicategory equipped with an object I, a pseudofunctor $\widetilde{\otimes}$, pseudonatural families of equivalences witnessing the weak associativity and unitality of $\widetilde{\otimes}$, and invertible modifications witnessing the axioms of a monoidal category.

We now make this precise, starting with the definition of equivalences. These generalize equivalences of categories.

Definition 6. *An* equivalence *between objects A and B in a bicategory $\mathscr{B}$ is a pair of 1-cells $f : A \to B$ and $f^\bullet : B \to A$ together with invertible 1-cells $f \circ f^\bullet \Rightarrow \mathsf{Id}_B$ and $\mathsf{Id}_A \Rightarrow f^\bullet \circ f$.*

A *pseudonatural equivalence* is a pseudonatural transformation in which each component has the structure of an equivalence; this induces an equivalence in the hom-bicategory.

The definition is now as advertised. To state it, we introduce some notation for the 2-cell diagrams—known as *pasting diagrams*—that we will use in the rest of the paper.

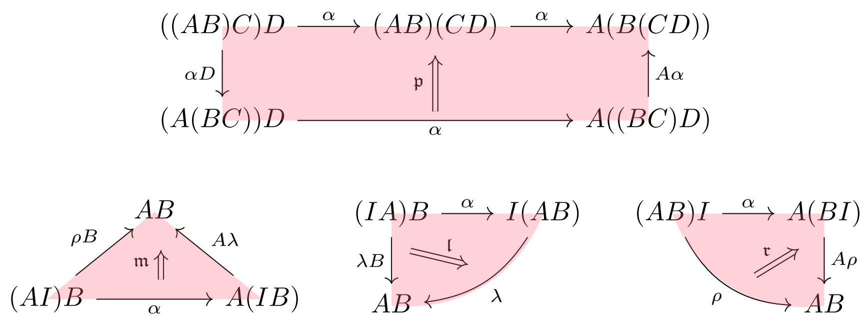


Figure 2: The structural modifications of a monoidal bicategory

Notation 1. *To save space and improve readability,*

- *We use juxtaposition for the tensor product, e.g.* $(AB)C$ *means* $(A \otimes B) \otimes C$*;*
- *We omit the subscripts on the components of pseudonatural transformations and modifications,* e.g. $\boldsymbol{m}$ *instead of* $\boldsymbol{m}_A$*;*
- *We use a subscript notation for the action of a pseudofunctor* T, e.g. T_{AT_B} *means* $T(A \otimes T(B))$.
- *We write* $\cong$ *for any pseudonaturality 2-cell as in* (2)*, and in equations we omit the arrows showing the directions of 2-cells. These labels can be inferred from the type.*

Definition 7 (*e.g.* [81])**.** *A* monoidal bicategory *is a bicategory* $\mathscr{B}$ *equipped with a pseudofunctor* $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ *and an object* $I \in \mathscr{B}$*, together with the following data:*

- *Pseudonatural equivalences* α, λ *and* ρ *with components* $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ *(the* associator*),* $\lambda_A : I \otimes A \to A$*, and* $\rho_A : A \otimes I \to A$ *(the* unitors*);*
- *Invertible modifications* $\mathfrak{p}, \mathfrak{l}, \mathfrak{m}$ *and* $\mathfrak{r}$ *with components shown in Figure 2, subject to coherence axioms.*

A symmetric *monoidal bicategory is a monoidal bicategory equipped with a pseudonatural equivalence* β *with components* $\beta_{A,B} : A \otimes B \to B \otimes A$*, called the* braiding*, and invertible modifications governing the possible shufflings of three objects and expressing the symmetry of the braiding, subject to coherence axioms.*

For example (see *e.g.* [81] for full details), the cartesian product on the category **Set** induces a monoidal structure on the bicategory **Span**(**Set**) introduced in Section 1.2. The pseudofunctor $\otimes$ is defined on objects as $A \otimes A' = A \times A'$, and for spans $A \leftarrow S \to B$ and $A' \leftarrow S' \to B'$ we take the component-wise product to obtain $A \times A' \leftarrow S \times S' \to B \times B'$.

We also record the outcome of our discussion above; this establishes a conjecture made in [6].

Example 6. *If* $(\mathbb{C}, \otimes, I)$ *is a symmetric monoidal category, this lifts to a symmetric monoidal structure on* **Para**$(\mathbb{C})$.

General point. The coherence axioms of a monoidal bicategory can be difficult to verify directly. However, in many cases of interest the monoidal structure is induced from a more fundamental construction, as in **Span**(**Set**) above. This gives a systematic method for constructing (symmetric) monoidal bicategories, described abstractly in [85].

2.4 Coherence theorems

As we have seen, bicategorical structures involve considerable data and many equations. Much of the difficulty, however, is tamed by various *coherence theorems*. These generally show that any two parallel 2-cells built out of the structural data are equal. Appropriate coherence theorems apply to bicategories [53] pseudofunctors [33], (symmetric) monoidal bicategories ([32, 34]) and pseudomonads [47].

We rely heavily on the coherence of bicategories and pseudofunctors when writing pasting diagrams of 2-cells: in particular we omit all compositors and unitors for pseudofunctors, and ignore the weakness of 1-cell composition. Thus, strictly speaking our diagrams do not type-check, but coherence guarantees the resulting 2-cell is the same no matter how one fills in the structural details. This is standard practice; for a detailed justification see *e.g.* [78, §2.2].

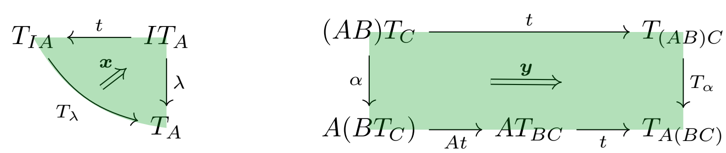
3 Strong pseudomonads

We follow the categorical setting by first saying what it means for a pseudo*functor* to be strong, then giving the additional data and axioms to make a pseudo*monad* strong. Section 3.3 contains many examples.

3.1 Strong pseudofunctors

For the moment we only consider strengths on the left. In all diagrams below we follow our Notation 1.

Definition 8. *Let $(\mathscr{B}, \otimes, I)$ be a monoidal bicategory. A* left strength *for a pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ is a pseudonatural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$, equipped with invertible modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ expressing the compatibility of t with the left unitor and the associator:*



These modifications must themselves be compatible with the monoidal structure, as per the three axioms of Figure 3.

Remark 1. *The three axioms of a strong pseudofunctor have a natural symmetry. Each diagram can be viewed as a cylinder, with the structural modifications of the monoidal bicategory (in pink) as the top and bottom. The axioms then stipulate how to pass these structural modifications through $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ from the top to the bottom of the cylinder.*

A left strength for a pseudofunctor T can be used to define a parametrised version of the functorial action: for any map $\Gamma \otimes X \to Y$ we can now define a map $\Gamma \otimes TX \to TY$. This suggests the following (recall Example 1 and Example 2).

Example 7. *If (F, t) is a strong functor on a symmetric monoidal category $(\mathbb{C}, \otimes, I)$, then the induced pseudofunctor $\widetilde{F}$ on* **Para**$(\mathbb{C})$ *is also strong. The pseudonatural transformation has components $\widetilde{t}_{A,B} := \widetilde{t_{A,B}}$; this has parameter I, so $\boldsymbol{x}$ and $\boldsymbol{y}$ are both of the form $I^{\otimes i} \xrightarrow{\cong} I^{\otimes j}$ for $i, j \in \mathbb{N}$.*

3.2 Strong pseudomonads

If a strong pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ is also a pseudomonad, then we must ask for additional data to relate the strength and the monad structure, and this data must be compatible with the modifications $\boldsymbol{x}, \boldsymbol{y}$ we already have.

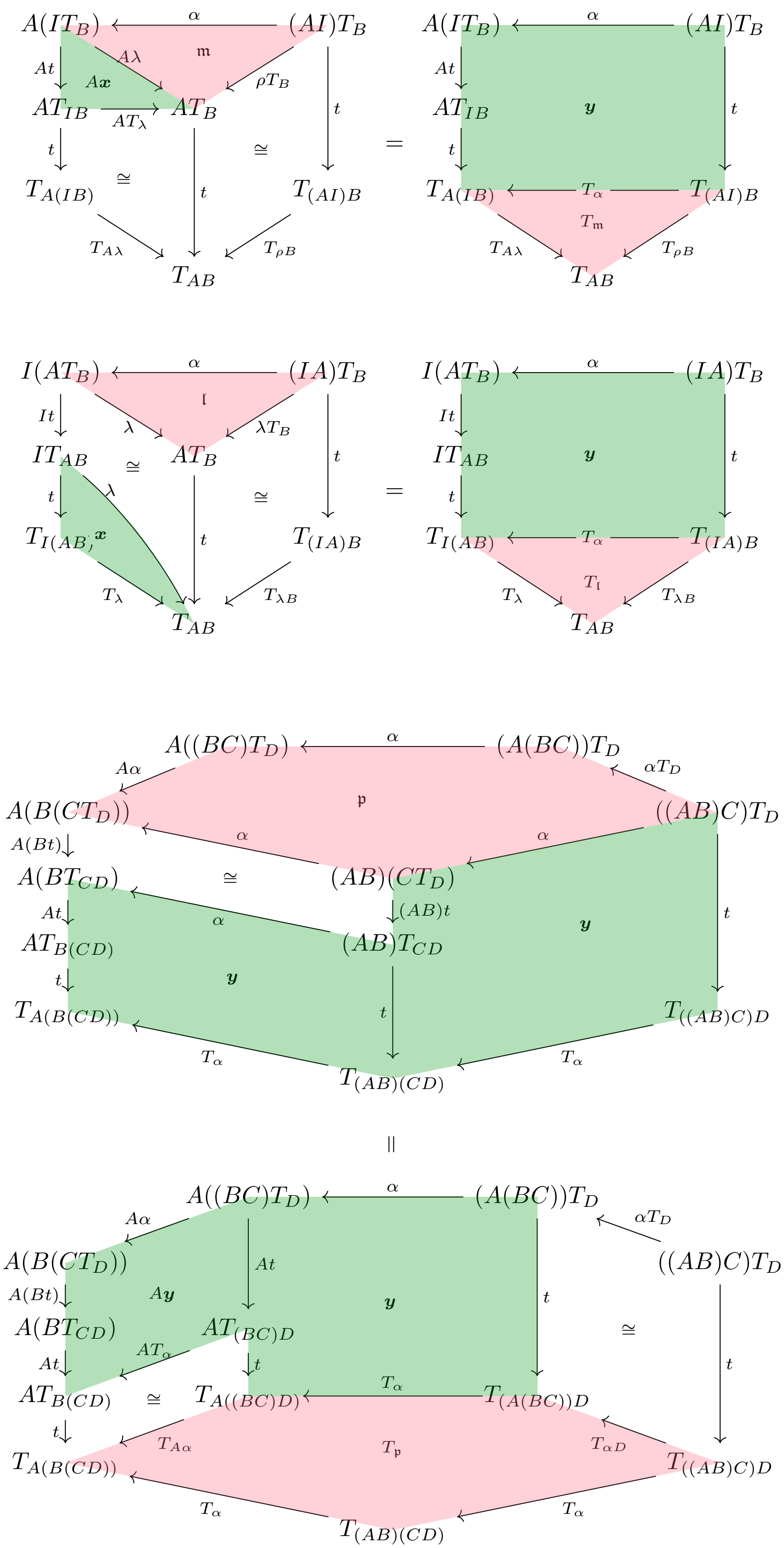
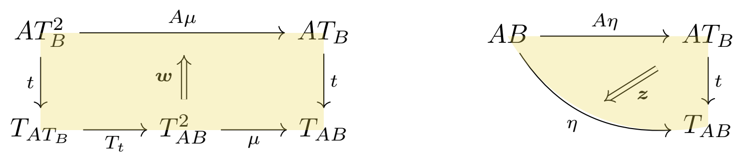


Figure 3: Coherence axioms for a strong pseudofunctor.

Definition 9. *Let* $(\mathscr{B}, \otimes, I)$ *be a monoidal bicategory. A* left strength *for a pseudomonad* (T, η, μ) *consists of a left strength* $(t, \boldsymbol{x}, \boldsymbol{y})$ *for the underlying pseudofunctor, together with invertible modifications*



expressing the compatibility of t *with the pseudomonad structure. This is subject to the coherence axioms of Figure 4, namely: three axioms expressing compatibility with the monad structure, two axioms laws relating* $\boldsymbol{x}$ *respectively with* $\boldsymbol{z}$ *and* $\boldsymbol{w}$*, and two axioms relating* $\boldsymbol{y}$ *respectively with* $\boldsymbol{z}$ *and* $\boldsymbol{w}$*.*

Extending Example 4 and Example 7, we obtain the following. The 2-cells $\boldsymbol{w}$ and $\boldsymbol{z}$ are defined similarly to $\boldsymbol{x}$ and $\boldsymbol{y}$.

Example 8. *A strong monad on a symmetric monoidal category* $(\mathbb{C}, \otimes, I)$ *determines a strong pseudomonad on* $\mathbf{Para}(\mathbb{C})$*.*

3.3 Examples of strong pseudomonads

In this section we show that several important classes of pseudomonad are strong in the way one would expect from the categorical setting; these help confirm that our definitions are correct. In each case the proof makes heavy use of the relevant coherence theorem for checking the axioms.

First, recall that if (M, m, e) is a monoid in a monoidal category $(\mathbb{C}, \otimes, I)$ then the functor $(-)\otimes M$ becomes a monad with unit and multiplication given via e and m (*c.f.* Example 3). This monad is canonically strong, with strength given by the canonical isomorphism $(A \otimes B) \otimes M \xrightarrow{\cong} A \otimes (B \otimes M)$. Also note that every monad T is strong with respect to the cocartesian structure $(0, +)$, with strength $[T\mathrm{inl} \circ \eta_A, T\mathrm{inr}] : A + TB \to T(A + B)$. These facts bicategorify. The bicategorical version of a monoid is called a *pseudomonoid* ([42, 15]), and every pseudomonoid defines a pseudomonad similarly to Example 3.

Lemma 1.

1. *For any pseudomonoid* (M, m, e, a, l, r) *on a monoidal bicategory* $(\mathscr{B}, \otimes, I)$ *the pseudomonad* $(-) \otimes M$ *has a strength given by the associator* α *of* $\otimes$*.*

2. *Every pseudomonad is canonically strong with respect to the cocartesian monoidal structure* $(+, 0)$*.*

Proof. For both claims, one constructs the data by following the corresponding 1-categorical argument and filling the commuting diagrams with the appropriate 2-cells; the equations hold by coherence. For (1), for instance, the structural modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ are given using $\mathfrak{l}$ and $\mathfrak{p}$, respectively, while $\boldsymbol{w}$ and $\boldsymbol{z}$ are given using $\mathfrak{r}$ and $\mathfrak{p}$, respectively, together with the pseudonaturality of $\alpha^{\bullet}$. The axioms hold by the coherence of pseudomonoids [47]. Similarly for (2): the strength has components $[T\mathrm{inl} \circ \eta_A, T\mathrm{inr}] : A + TB \to T(A + B)$ and the structural modifications are given by taking the categorical proof and filling in the commuting diagrams with the appropriate 2-cells. The equations hold by coherence for bicategories with finite products [69] and the fact all the structural 2-cells are invertible. □

In particular, a pseudomonoid in $(\mathbf{Cat}, \times, 1)$ is a monoidal category so Lemma 1(1) applies to the Writer pseudomonad (Example 3). We can also use this lemma to derive a result about pseudomonads on spans. For any category $\mathbb{C}$ with pullbacks there exists a bicategory of spans $\mathbf{Span}(\mathbb{C})$ similar to that defined in Section 1.2 for $\mathbf{Set}$. For $\mathbb{C} := \mathbf{Set}$, or more generally any *lextensive* category [7], the bicategory $\mathbf{Span}(\mathbb{C})$ has finite biproducts—bicategorical products and coproducts which coincide—by [49, Theorem 6.2]. Moreover, by [35, Corollary A.4], every *cartesian monad* (monad for which the underlying functor preserves pullbacks, and such that every naturality

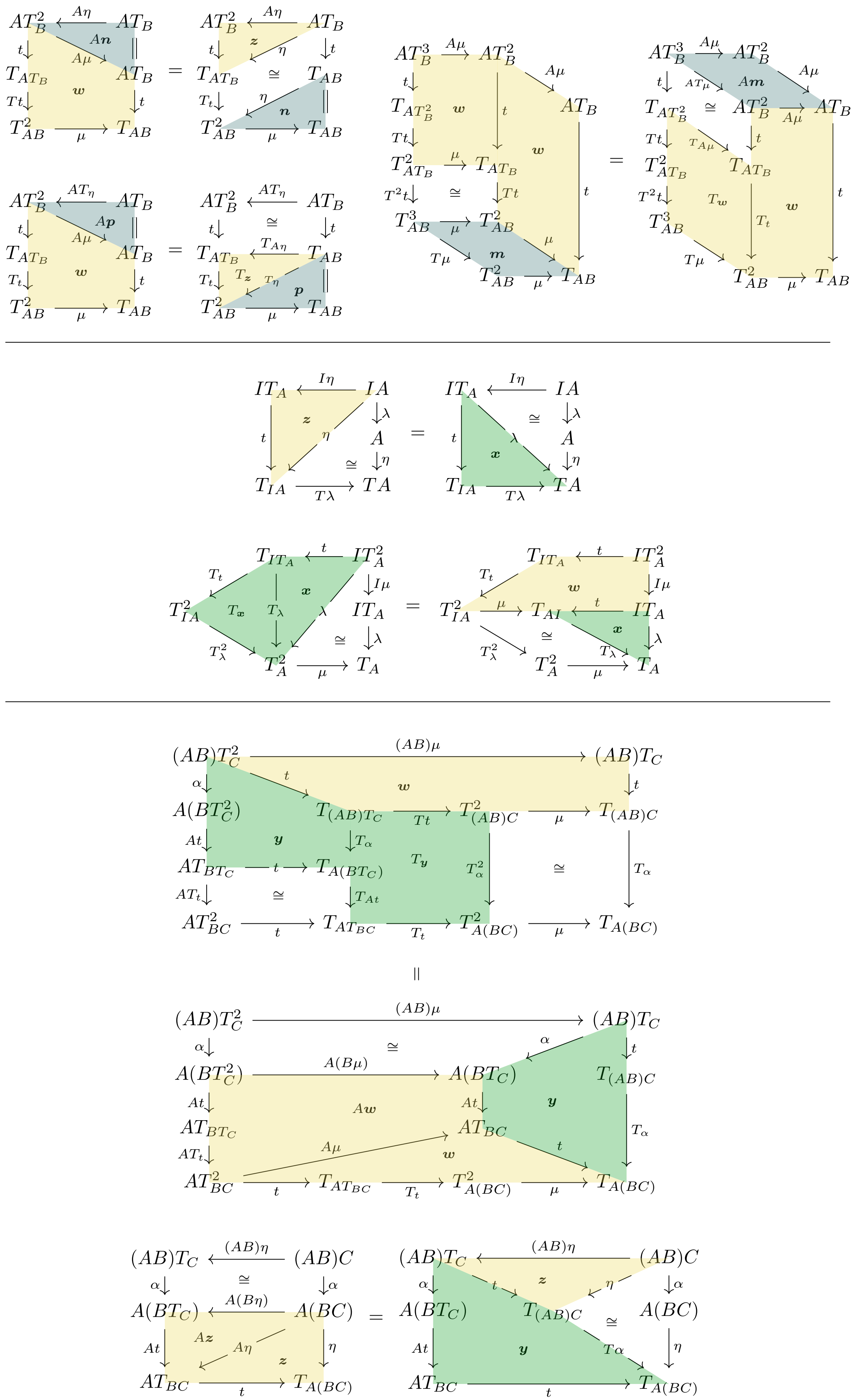


Figure 4: Coherence axioms for a strong pseudomonad.

square for μ and η is a pullback square) lifts to a pseudomonad on $\mathbf{Span}(\mathbb{C})$. So we have the following.

Corollary 1. *Any cartesian monad on a lextensive category $\mathbb{C}$ (such as* **Set***) lifts to a strong pseudomonad on* $\mathbf{Span}(\mathbb{C})$

The next example covers two cases of importance in the semantics of programming languages. The proof follows essentially immediately from the corresponding categorical facts and the particularly strong form of coherence enjoyed by cartesian closed bicategories (see [22, Principle 1.3]).

Lemma 2. *For any cartesian closed bicategory (see* e.g. *[21]) $(\mathscr{B}, \times, 1, \Rightarrow)$ and objects $S, R \in \mathscr{B}$, there exist strong pseudomonads $S \Rightarrow (S \times -)$ (the* state *pseudomonad) and $(- \Rightarrow R) \Rightarrow R$ (the* continuation *pseudomonad).*

For our final class of examples, recall that every functor F on **Set** is canonically strong with respect to the cartesian structure, with $t_{A,B} : A \times FB \to F(A \times B)$ defined by $t_{A,B}(a, w) := F(\lambda b \,.\, \langle a, b \rangle)(w)$, and moreover that the same construction makes every monad on **Set** strong [64, Proposition 3.4]. A similar fact holds for bicategories.

Proposition 1. *Every pseudofunctor (resp. pseudomonad) on $(\mathbf{Cat}, \times, 1)$ has a canonical choice of strength.*

Proof sketch. Similarly to the categorical proof, for any pseudofunctor $F : \mathbf{Cat} \to \mathbf{Cat}$, and $a \in \mathbb{A}$ one has $F(\lambda b \,.\, \langle a, b \rangle) : F(\mathbb{B}) \to F(\mathbb{A} \times \mathbb{B})$. However, since F is now a pseudofunctor we also have a natural transformation $F(\lambda b \,.\, \langle f, b \rangle)$ for each $f : a \to a'$ in $\mathbb{A}$, with components $F(\lambda b \,.\, \langle f, b \rangle)_w : F(\lambda b \,.\, \langle a, b \rangle) \to F(\lambda b \,.\, \langle a', b \rangle)$ in $F(\mathbb{A} \times \mathbb{B})$. We may therefore define a functor $t_{\mathbb{A},\mathbb{B}} : \mathbb{A} \times F\mathbb{B} \to F(\mathbb{A} \times \mathbb{B})$ sending a pair of objects (a, w) to $F(\lambda b \,.\, \langle a, b \rangle)(w)$ and a pair of morphisms $(a \xrightarrow{f} a', w \xrightarrow{g} w')$ to the composite $F(\lambda b \,.\, \langle a', b \rangle)(g) \circ F(\lambda b \,.\, \langle f, b \rangle)_w$. This is functorial because F is functorial on natural transformations and $F(\lambda b \,.\, \langle a, b \rangle)$ is a functor, and pseudonatural via the compositor for F.

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined using the compositor and unitor for F, and the coherence of pseudomonads ensures the axioms hold. Finally, if T is a pseudomonad then one defines $\boldsymbol{w}$ and $\boldsymbol{z}$ using the pseudonaturality of η and μ: this is similar to the proof in the categorical setting, where one uses the naturality of the unit and multiplication to show the two compatibility laws hold. Again, the axioms follow from coherence. □

In summary, there are many natural examples of strong pseudomonad. Beyond the general examples we have presented, there are concrete examples arising in denotational semantics, *e.g.* linear continuation pseudomonads in dialogue categories [61], to be studied independently.

4 Actions of monoidal bicategories

In this section we justify our definition of strength (Definition 9) by showing a bicategorical version of a well-known correspondence theorem (*e.g.* [57, Proposition 4.3]). We show that to give a left strength for a pseudomonad T on $(\mathscr{B}, \otimes, I)$ is to give a left action of $(\mathscr{B}, \otimes, I)$ on the Kleisli bicategory $\mathscr{B}_T$ that extends the monoidal structure.

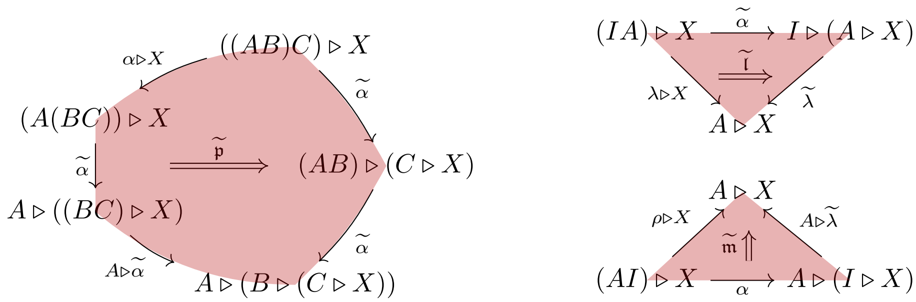
We start by defining actions of monoidal bicategories. We are not aware of a definition in the literature, but it is clear what it ought to be. As observed in [39], a left action on a category is equivalently a bicategory with two objects and certain hom-categories taken to be trivial. We therefore define a left action on a bicategory so it is equivalently a *tricategory* (see [32]) with two objects and certain hom-bicategories taken to be trivial. It follows from the coherence of tricategories ([32, 33]) that every diagram of 2-cells constructed using the structural data of an action must commute. More explicitly:

Definition 10. *A* left action *of a monoidal bicategory* $(\mathscr{B}, \otimes, I)$ *on a bicategory* $\mathscr{C}$ *consists of a pseudo-functor* $\triangleright : \mathscr{B} \times \mathscr{C} \to \mathscr{C}$*, together with the following data:*

- *Pseudonatural equivalences* $\widetilde{\lambda}$ *and* $\widetilde{\alpha}$ *with components*

$$\widetilde{\lambda}_X : I \triangleright X \to X \qquad \widetilde{\alpha}_{A,B,X} : (A \otimes B) \triangleright X \to A \triangleright (B \triangleright X);$$

- *Invertible modifications as shown below, satisfying the same coherence axioms as* $\mathfrak{p}, \mathfrak{m}$*, and* $\mathfrak{l}$ *in a monoidal bicategory (*e.g. *[81]):*



The tensor product $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ on a monoidal bicategory determines an action of $\mathscr{B}$ on itself, with all the structure given by the monoidal data. We call this the *canonical action* of a monoidal bicategory on itself. Every strong pseudomonad also induces an action.

Proposition 2. *Every strong pseudomonad* (T, t) *on* $(\mathscr{B}, \otimes, I)$ *induces an action of* $\mathscr{B}$ *on the Kleisli bicategory* $\mathscr{B}_T$*, where the pseudofunctor* $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ *is given on objects by* $A \triangleright B = A \otimes B$*, and on morphisms as*

$$f \triangleright g \quad := \quad \left(A \otimes B \xrightarrow{f \otimes g} A' \otimes TB' \xrightarrow{t} T(A' \otimes B')\right)$$

for $f : A \to A'$ *and* $g : B \to TB'$*.*

Proof sketch. The action on 2-cells is the same as that on morphisms. The compositor and unitor for $\triangleright$ are given by the modifications $\boldsymbol{w}$ and $\boldsymbol{z}$ that come with the strength. We use $\boldsymbol{x}$ and $\boldsymbol{y}$ to construct the strength data $\widetilde{\lambda}$ and $\widetilde{\alpha}$ from the monoidal data λ and α, and finally we use $\boldsymbol{z}$ again to lift $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ to $\widetilde{\mathfrak{p}}, \widetilde{\mathfrak{m}}, \widetilde{\mathfrak{l}}$. The strength axioms ensure that this forms an action. □

The action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ induced by a strong pseudomonad can be seen as an extension of the canonical action $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ given by the monoidal structure. Indeed, we have a pseudonatural transformation

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{B}_T & \xrightarrow{\ \triangleright\ } & \mathscr{B}_T \\ {\scriptstyle \mathscr{B} \times K}\uparrow & \overset{\theta}{\Longrightarrow} & \uparrow{\scriptstyle K} \\ \mathscr{B} \times \mathscr{B} & \xrightarrow[\otimes]{} & \mathscr{B} \end{array} \tag{4}$$

where $K : \mathscr{B} \to \mathscr{B}_T$ is the identity-on-objects pseudofunctor which sends $f : A \to A'$ to $\eta_{A'} \circ f : A \to TA'$. Moreover, the two actions $\triangleright$ and $\otimes$ agree on objects, and the 1-cell components $\theta_{A,B}$ of the transformation are all the identity. Such a transformation is known as an *icon* [48]. The 2-cell components of θ are nontrivial: for each $f : A \to A'$ and $g : B \to B'$ we have an isomorphism

$$\theta_{f,g} : f \triangleright K(g) \overset{\cong}{\Longrightarrow} K(f \otimes g)$$

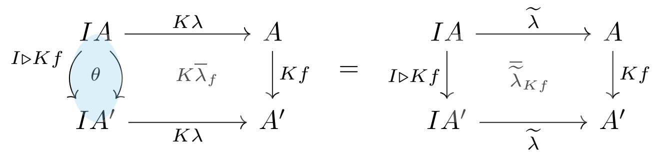
derived from the modification z, satisfying the coherence laws.

4.1 A correspondence theorem

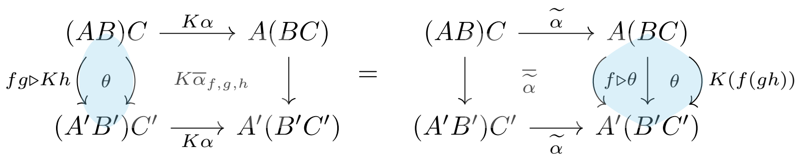
We now prove an equivalence between left strengths and left actions, based on the following definition.

Definition 11. *For a pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, an* extension of the canonical action of $\mathscr{B}$ on itself *is a pair $(\triangleright, \theta)$ consisting of a left action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ and an icon θ as in (4) such that the structural data $\widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{m}}, \widetilde{\mathfrak{p}}$ for the action extends the corresponding monoidal data via K and θ. Explicitly, for all objects A, B, C, D and morphisms $f : A \to A'$, $g : B \to B'$, and $h : C \to C'$ in $\mathscr{B}$:*

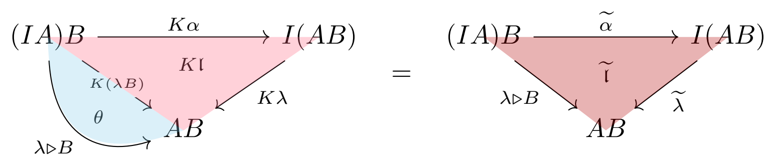
1. $\widetilde{\lambda}_A = K\lambda_A$, *and*



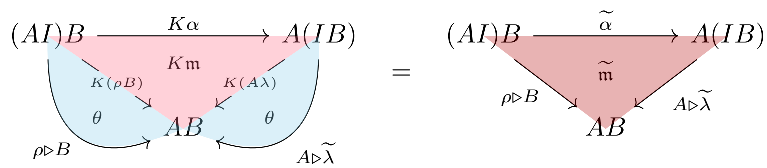
2. $\widetilde{\alpha}_{A,B,C} = K\alpha_{A,B,C}$, *and*



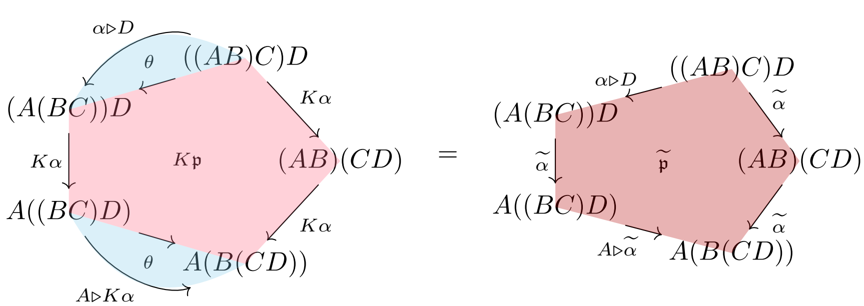
3.



4.



5.



Remark 2. *Just as we have defined an action to be a tricategory with two objects and certain data taken to be trivial, so one may alternatively present the preceding definition as a* trihomomorphism *(morphism of tricategories [32]) with certain data taken to be trivial. This follows the tradition for defining structure for monoidal bicategories, such as monoidal pseudofunctors, as 'one-object' versions of the corresponding tricategorical structure (*c.f. *[9]).*

Our correspondence theorem uses the following two categories for a pseudomonad T on $(\mathscr{B}, \otimes, I)$:

- **LeftStr**(T), the category whose objects are left strengths for T, and whose morphisms from t to t' are modifications which commute with all the strength data;
- **LeftExt**(T), the category whose objects are extensions of the canonical action of $\mathscr{B}$ on itself, and whose morphisms from $(\triangleright, \theta)$ to $(\triangleright', \theta')$ are icons $\triangleright \Rightarrow \triangleright'$ which commute with θ and θ'.

Theorem 1. *For any pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the categories* **LeftStr**(T) *and* **LeftExt**(T) *are equivalent.*

Proof notes. The proof follows the categorical construction (see [57, Prop. 4.3]). For every left strength t for T, the induced action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ extends the canonical action of $\mathscr{B}$ on itself, by construction. Conversely, any extension $(\triangleright, \theta)$ induces a strength $t_{A,B} = \mathrm{id}_A \triangleright \mathrm{id}_{TB}$, where id_{TB} is regarded as an element of $\mathscr{B}_T(TB, B)$. These constructions are inverses, up to isomorphism, as we verify directly. We then verify the axioms. In each direction, there is a tight match-up between the equations of the given structure and the equations for the required structure; for an outline, see Table 1. □

Thus, actions and strengths are related in the bicategorical setting just as they are in the categorical one. This provides further justification for the coherence axioms of Definition 9.

5 Pseudomonads with a strength on each side

Categorically, it is often the case that a monad T supports a strength on both sides, and the two strengths are compatible: T is then called *bistrong* (see *e.g.* [57]). This is the case, for instance, if T has left strength t and the underlying category is symmetric monoidal, because we can construct a right strength using the left strength and the symmetry β, as follows:

$$T(A) \otimes B \xrightarrow{\beta} B \otimes T(A) \xrightarrow{t} T(B \otimes A) \xrightarrow{T\beta} T(A \otimes B). \tag{5}$$

We now bicategorify these results. As well as providing an important sanity-check for our definitions, this covers the situation most common in semantics. In Section 6 we shall see that having two compatible strengths is a requirement for the Kleisli bicategory of a pseudomonad T to be premonoidal.

Strength	Action
Axioms for a strong pseudofunctor (Fig. 3)	Modification axioms for the 2-cells $\widetilde{\mathfrak{m}}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{p}}$ determined by Def. 11 (3, 4, 5)
Compatibility between $\mathbf{m}, \mathbf{n}, \mathbf{p}$ and $\mathbf{z}, \mathbf{w}$ (Top row in in Fig. 4)	Pseudofunctor axioms for $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$
Compatibility between $\mathbf{x}$ and $\mathbf{z}, \mathbf{w}$ (Middle row in Fig. 4)	Pseudonaturality of the transformation $\widetilde{\lambda}$ determined by Def. 11(1)
Compatibility between $\mathbf{y}$ and $\mathbf{z}, \mathbf{w}$ (Bottom row of axioms in Fig. 4)	Pseudonaturality of the transformation $\widetilde{\alpha}$ determined by Def. 11($\times$2)

Table 1: Relating the data and equations on each side of the correspondence in Theorem 1.

5.1 Compatible left and right strengths

A *right strength* for a pseudomonad consists of a pseudonatural transformation $s_{A,B} : TA \otimes B \to T(A \otimes B)$ equipped with four invertible modifications analogous to x, y, z, w.

Informally, a left strength $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and a right strength $s_{A,B} : TA \otimes B \to T(A \otimes B)$ are *compatible* if parameters on each side can be passed through T in any order. Categorically, one makes this precise by asking that the two obvious maps $(A \otimes TB) \otimes C \to T(A \otimes (B \otimes C))$ are equal. For the bicategorical definition, we replace this equation by a coherent isomorphism.

Definition 12. *A* bistrong *pseudomonad on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ is a pseudomonad T equipped with a left strength t and a right strength s, and an invertible modification*

$$\begin{array}{ccccc} & (AT_B)C & \xrightarrow{\alpha} & A(T_BC) & \\ {\scriptstyle tC}\swarrow & & & & \searrow{\scriptstyle As} \\ T_{AB}C & & \overset{b}{\Longrightarrow} & & AT_{BC} \\ {\scriptstyle s}\searrow & & & & \swarrow{\scriptstyle t} \\ & T_{(AB)C} & \xrightarrow[T_\alpha]{} & T_{A(BC)} & \end{array}$$

satisfying the two axioms in Figure 5.

This definition is sufficient to recover the categorical situation: if $(\mathscr{B}, \otimes, I)$ is symmetric monoidal and (T, t) is a left-strong pseudomonad, then the composite pseudonatural transformation with components as in (5) can always be given the structure of a right strength for T.

Proposition 3. *Every left-strong pseudomonad on a symmetric monoidal bicategory is bistrong in a canonical way.*

5.2 Towards commutative pseudomonads

A bistrong monad on a monoidal category is called *commutative* if the two obvious maps $T(A) \otimes T(B) \to T(A \otimes B)$ are equal ([45, 46]). In computational terms, commutativity means that we can re-order program statements. Some well-known side-effects (probability, divergence) are commutative, while many others (continuations, state) are not.

A promising starting point for commutativity would be the notion of *pseudo-commutative 2-monad* [38]. Thus, a commutative pseudomonad should be a bistrong pseudomonad (T, t, s) with an invertible modification

$$\begin{array}{ccccc} T_AT_B & \xrightarrow{s} & T_{AT_B} & \xrightarrow{Tt} & T^2_{AB} \\ {\scriptstyle t}\downarrow & & \overset{\gamma}{\Longrightarrow} & & \downarrow{\scriptstyle \mu} \\ T_{T_AB} & \xrightarrow[Ts]{} & T^2_{AB} & \xrightarrow[\mu]{} & T_{AB} \end{array}$$

representing the commutativity, satisfying a number of coherence properties such as those of [38, §3.2]. We leave the validation of this definition as a topic for further study.

6 Premonoidal bicategories

Finally, we introduce premonoidal bicategories; we then validate this definition by showing the Kleisli bicategory of a bistrong pseudomonad is premonoidal (Section 6.4).

6.1 Premonoidal categories

Premonoidal categories axiomatize the semantics of effectful languages more directly than strong monads, in the sense that a program $(\Gamma \vdash P : A)$ is interpreted as a morphism $\Gamma \to A$. For such a model to be sound, one must deal with the fact that statements in effectful programs cannot generally be re-ordered (recall Figure 1). So if we have another program $(\Delta \vdash Q : B)$, there are two ways of interpreting the pair $\big(\Gamma \otimes \Delta \vdash (P, Q) : A \otimes B\big)$, depending on whether we run P or Q

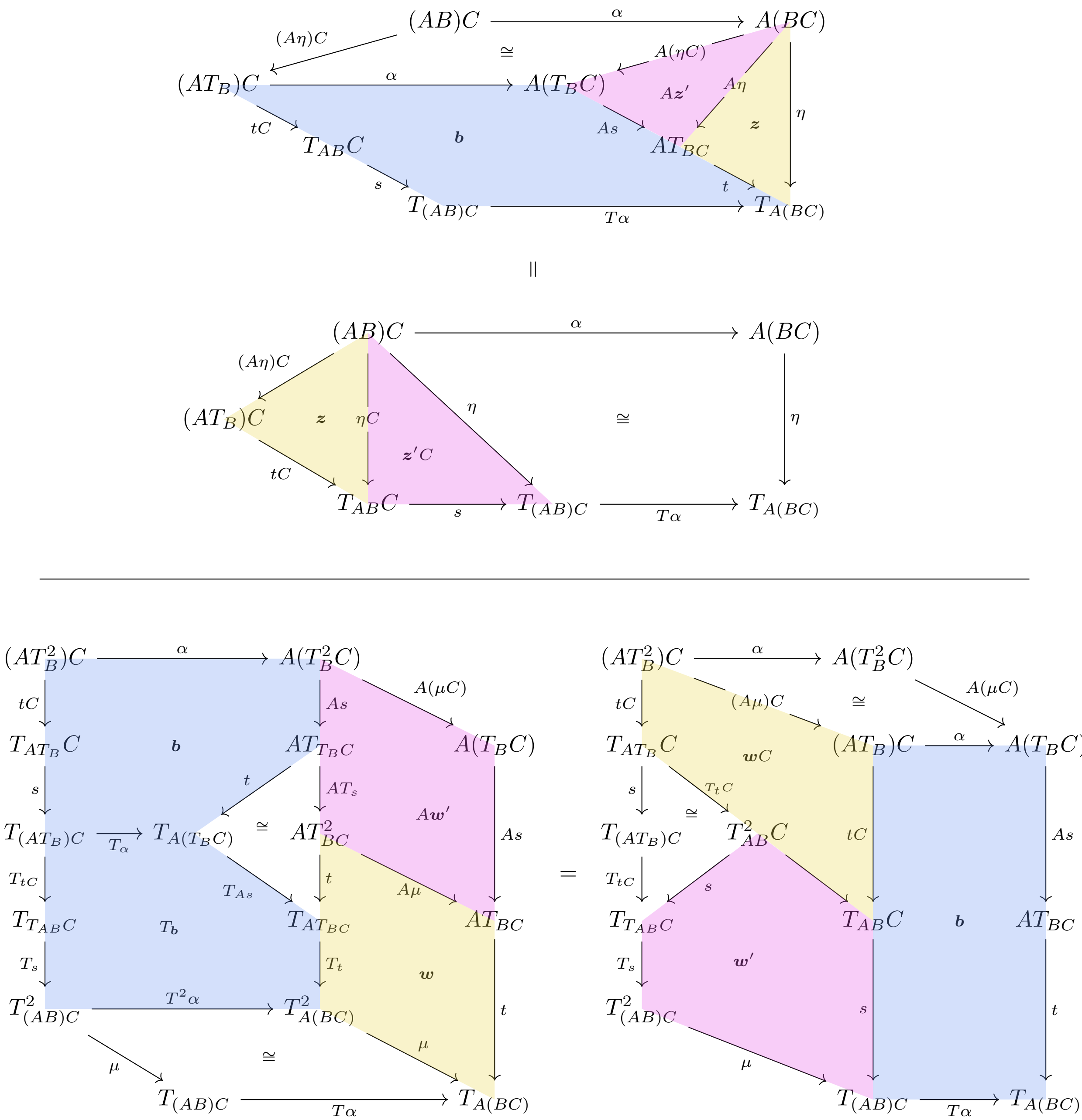


Figure 5: Coherence axioms for a bistrong pseudomonad.

first. As a consequence, $\otimes$ is not a functor of two arguments. In this section we briefly recall the formal definition and the premonoidal structure of a key example, namely the Kleisli category of a bistrong monad.

First, some preliminaries. A *binoidal category* is a category $\mathbb{D}$ equipped with functors

$$A \rtimes (-) : \mathbb{D} \longrightarrow \mathbb{D} \qquad\qquad (-) \ltimes B : \mathbb{D} \longrightarrow \mathbb{D}$$

for every $A, B \in \mathbb{D}$, such that $(A \rtimes -)(B) = (- \ltimes B)(A)$. We write $A \otimes B$, or just AB, for their joint value. (The notation is intended to suggest that $\rtimes$ is a 'complete' functor only on the right.) A map $f : A \to A'$ is *central* if the two diagrams

$$\begin{array}{ccc} AB & \xrightarrow{A \rtimes g} & AB' \\ {\scriptstyle f \ltimes B}\downarrow & & \downarrow{\scriptstyle f \ltimes B'} \\ A'B & \xrightarrow[A' \rtimes g]{} & A'B' \end{array} \qquad\qquad \begin{array}{ccc} BA & \xrightarrow{g \ltimes A} & B'A \\ {\scriptstyle B \rtimes f}\downarrow & & \downarrow{\scriptstyle B' \rtimes f} \\ BA' & \xrightarrow[g \ltimes A']{} & B'A' \end{array} \tag{6}$$

commute for every $g : B \to B'$. Semantically, f corresponds to a program which may be run at any point without changing the observable result. We can now give the definition.

Definition 13 ([73])**.** *A* premonoidal category *is a binoidal category* $(\mathbb{D}, \rtimes, \ltimes)$ *equipped with a unit object* I *and central isomorphisms* $\rho_A : AI \to A$*,* $\lambda_A : IA \to A$ *and* $\alpha_{A,B,C} : (AB)C \to A(BC)$ *for every* $A, B, C \in \mathbb{D}$*, satisfying the triangle and pentagon axioms for a monoidal category and the five following naturality conditions:*

$$\begin{array}{ccc} (AB)C & \xrightarrow{(f \ltimes B) \ltimes C} & (A'B)C \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A(BC) & \xrightarrow[f \ltimes (BC)]{} & A'(BC) \end{array} \quad \begin{array}{ccc} (AB)C & \xrightarrow{(A \rtimes g) \ltimes C} & (AB')C \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A(BC) & \xrightarrow[A \rtimes (g \ltimes C)]{} & A(B'C) \end{array} \quad \begin{array}{ccc} (AB)C & \xrightarrow{(AB) \rtimes h} & (AB)C' \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A(BC') & \xrightarrow[A \rtimes (B \rtimes h)]{} & A(BC') \end{array}$$

$$\begin{array}{ccc} IA & \xrightarrow{I \rtimes f} & IA' \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ A & \xrightarrow[f]{} & A' \end{array} \qquad\qquad \begin{array}{ccc} AI & \xrightarrow{f \ltimes I} & A'I \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ A & \xrightarrow[f]{} & A' \end{array}$$

Notice that α cannot be a natural transformation in all arguments simultaneously, because $\otimes$ is not a functor on $\mathbb{C}$.

One motivating example for premonoidal categories is the Kleisli category of a bistrong monad (T, t, s) on a monoidal category $(\mathbb{C}, \otimes, I)$, where for every A, B, f and g we have

$$\begin{aligned} A \rtimes g &:= \big(AB \xrightarrow{A \otimes g} AT_{B'} \xrightarrow{t} T_{AB'}\big) \\ f \ltimes B &:= \big(AB \xrightarrow{f \otimes B} T_{A'}B \xrightarrow{s} T_{A'B}\big). \end{aligned} \tag{7}$$

The structural data is lifted from $\mathbb{C}$. Premonoidal categories are therefore a clean axiomatization of the monadic semantics, which also captures first-order effectful languages whose syntax does not explicitly involve a monad (*e.g.* [63, 74]).

6.2 Binoidal bicategories and centrality

We now turn to the bicategorical setting. Following the general bicategorical pattern, centrality for 1-cells will be extra data, rather than a property. This raises new subtleties, notably because it affects the type of the structural modifications.

Definition 14. *A* binoidal bicategory *is a bicategory* $\mathscr{B}$ *equipped with pseudofunctors* $A \rtimes (-)$ *and* $(-) \ltimes B$ *for every* $A, B \in \mathscr{B}$*, such that* $A \rtimes B = A \ltimes B$*.*

We denote the structure by $(\mathscr{B}, \rtimes, \ltimes)$, and as before we write $A \otimes B$, or just AB, for the joint value on objects.

To define centrality, observe that saying the diagrams in (6) commute for every g is exactly saying that the families of maps $f \ltimes B$ and $B \rtimes f$ are natural in B.

Definition 15 (Central 1-cell)**.** *A* central 1-cell *in a binoidal bicategory* $(\mathscr{B}, \rtimes, \ltimes)$ *is a 1-cell* $f : A \to A'$ *with 2-cells*

$$\begin{array}{ccc} AB & \xrightarrow{A \rtimes g} & AB' \\ {\scriptstyle f \ltimes B}\downarrow & \overset{\mathsf{lc}^f_g}{\Longleftarrow} & \downarrow{\scriptstyle f \ltimes B'} \\ A'B & \xrightarrow[A' \rtimes g]{} & A'B' \end{array} \qquad\qquad \begin{array}{ccc} BA & \xrightarrow{g \ltimes A} & B'A \\ {\scriptstyle B \rtimes f}\downarrow & \overset{\mathsf{rc}^f_g}{\Longleftarrow} & \downarrow{\scriptstyle B' \rtimes f} \\ BA' & \xrightarrow[g \ltimes A']{} & B'A' \end{array} \tag{8}$$

for every $g : B \to B'$*, forming pseudonatural transformations*

$$(A \rtimes -) \xRightarrow{f \rtimes -} (A' \rtimes -) \qquad \textit{and} \qquad (- \ltimes A) \xRightarrow{- \ltimes f} (- \ltimes A')$$

respectively.

Definition 16 (Central 2-cell)**.** *A* central 2-cell σ *between central 1-cells* $(f, \mathsf{lc}^f, \mathsf{rc}^f)$ *and* $(f', \mathsf{lc}^{f'}, \mathsf{rc}^{f'})$ *is a 2-cell* $\sigma : f \Rightarrow f'$ *such that the 2-cells* $\sigma \ltimes B$ *and* $B \rtimes \sigma$ *(for* $B \in \mathscr{B}$*) define modifications* $\mathsf{lc}^f \Rrightarrow \mathsf{lc}^{f'}$ *and* $\mathsf{rc}^f \Rrightarrow \mathsf{rc}^{f'}$*, respectively.*

Every monoidal bicategory $(\mathscr{B}, \otimes, I)$ has a canonical binoidal structure, with $A \rtimes (-) := A \otimes (-)$ and $(-) \ltimes B := (-) \otimes B$. Every 1-cell in $\mathscr{B}$ canonically determines a central 1-cell because for any f and g we can set lc^f_g and $(\mathsf{rc}^g_f)^{-1}$ both to be $\mathsf{int}_{f,g}$, defined using the compositor for $\otimes$ as follows:

$$\mathsf{int}_{f,g} \quad := \quad \begin{array}{ccc} AB & \xrightarrow{A \otimes g} & AB' \\ {\scriptstyle f \otimes B}\downarrow & \overset{\phi}{\searrow}{\scriptstyle f \otimes g}\overset{\phi}{} & \downarrow{\scriptstyle f \otimes B'} \\ A'B & \xrightarrow[A' \otimes g]{} & A'B' \end{array} \tag{9}$$

By the functoriality of $\otimes$, every 2-cell becomes central.

For any bistrong pseudomonad (T, s, t) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the Kleisli bicategory $\mathscr{B}_T$ is binoidal: the pseudofunctors $\rtimes$ and $\ltimes$ are given as in (7), and every pure 1-cell $\widetilde{f} := \eta \circ f : A \to TA'$ becomes central with $\mathsf{lc}^{\widetilde{f}}$ and $\mathsf{rc}^{\widetilde{f}}$ constructed using structural data.

6.3 Premonoidal bicategories

As indicated above, the definition of premonoidal bicategory involves some subtleties not present in the categorical setting, particularly around the 2-cells. We begin by outlining these issues and how we deal with them.

Pseudonatural associators and unitors. In Definition 13 the associator α can only be natural in each argument separately because $\otimes$ is not a functor of two arguments. Similarly, we can only ask that the associator is pseudonatural in each argument separately. This entails asking for three pseudonatural transformations: one for each of the three naturality diagrams for α in Definition 13. Precisely, we ask that for every $A, B, C \in \mathscr{B}$ and f, g, h as in that definition, we get 2-cells $\overline{\alpha}_{f,B,C}$, $\overline{\alpha}_{A,g,C}$ and $\overline{\alpha}_{A,B,h}$ giving pseudonatural transformations

$$\begin{aligned} (\alpha_{-,B,C}, \overline{\alpha}_{-,B,C}) &: (- \ltimes B) \ltimes C \Rightarrow (-) \ltimes (B \otimes C) \\ (\alpha_{A,-,C}, \overline{\alpha}_{A,-,C}) &: (A \rtimes -) \ltimes C \Rightarrow A \rtimes (- \ltimes C) \\ (\alpha_{A,B,-}, \overline{\alpha}_{A,B,-}) &: (A \otimes B) \rtimes (-) \Rightarrow A \rtimes (B \rtimes -) \end{aligned} \tag{10}$$

We also need to give unitors ρ and λ, but for these pseudonaturality is straightforward since there is only one argument.

Structural modifications in each argument. The main remaining work is in giving the structural modifications. Just as in the case of α above, we cannot ask for modifications $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ and $\mathfrak{r}$ as we do for a monoidal bicategory: instead we must ask for families of 2-cells satisfying the modification condition in each component separately.

This introduces a new issue. In a monoidal bicategory, the fact that the structural modifications are modifications in each argument relies on $\otimes$ being a pseudofunctor of two arguments. For example, the type of $\mathfrak{l}$ (see Figure 2) uses the pseudonatural transformation with components $\lambda_A \otimes B : (IA)B \to AB$. For $g : B \to B'$, the 2-cell witnessing pseudonaturality of this transformation is the interchange isomorphism $\mathsf{int}_{\lambda_A, g}$ defined in (9). This 2-cell does not exist in a premonoidal bicategory, so instead we must use the centrality witness $\mathsf{lc}_g^{\lambda_A}$ for λ_A. Thus, we define $\mathfrak{l}$ to be a family of 2-cells $\mathfrak{l}_{A,B} : (\lambda_A \ltimes B) \Rightarrow \lambda_{A \otimes B} \circ \alpha_{I,A,B}$ that is a modification in each argument, in the sense that for every $A, B \in \mathscr{B}$ we get modifications in $\mathrm{Hom}(\mathscr{B}, \mathscr{B})$ as follows:

$$\begin{array}{ccc} (I \rtimes -) \rtimes B & \xrightarrow{\lambda \ltimes B} & (- \ltimes B) \\ & \searrow^{\alpha_{I,-,B}} \quad \mathfrak{l}_{-,B} \quad \nearrow_{\lambda_{-\ltimes B}} & \\ & I \rtimes (- \ltimes B) & \end{array} \qquad \begin{array}{ccc} (IA) \rtimes (-) & \xrightarrow{\mathsf{lc}^{\lambda}} & (A \rtimes -) \\ & \searrow^{\alpha_{I,A,-}} \quad \mathfrak{l}_{A,-} \quad \nearrow_{\lambda_{A\rtimes -}} & \\ & I \rtimes (A \ltimes -) & \end{array}$$

Notice that the left-hand diagram is exactly as in the definition of a monoidal bicategory: no adjustments are necessary because each transformation is pseudonatural in the open argument.

We repeat this process for each of the four structural modifications to obtain our definition of premonoidal bicategory: Figure 6 gives the diagrams that are not exactly as in the definition of a monoidal bicategory. (To save space we suppress the $\rtimes$ and $\ltimes$: these can be inferred from the bracketing.) Although we have changed the conditions for $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ and $\mathfrak{r}$ to be modifications, the equations of a monoidal bicategory are still well-typed. We also require only that the 1-cell data is central, because the centrality of 2-cells such as $\overline{\alpha}_{f,B,C}$ can be derived in the cases of interest: see Proposition 6.

Premonoidal bicategories. We give the main definition. Despite the subtleties outlined above, the passage from premonoidal categories is very natural.

Definition 17. *A* premonoidal bicategory *is a binoidal bicategory* $(\mathscr{B}, \rtimes, \ltimes)$ *with a* unit $I \in \mathscr{B}$ *and the following data:*

1. *Pseudonatural equivalences* $\lambda : I \rtimes (-) \Rightarrow \mathrm{id}$ *and* $\rho : (-) \ltimes I \Rightarrow \mathrm{id}$ *with each 1-cell component central;*

2. *An central 1-cell* $\alpha_{A,B,C}$ *for every* $A, B, C \in \mathscr{B}$*, with pseudonatural data as in* (10)*, and such that each component* $\alpha_{A,B,C}$ *is an equivalence;*

3. *For each* $A, B, C, D \in \mathscr{B}$*, invertible 2-cells* $\mathfrak{p}_{A,B,C,D}, \mathfrak{m}_{A,B}, \mathfrak{l}_{A,B}$ *and* $\mathfrak{r}_{A,B}$*, such that these form modifications in each argument as in Figure 6 or, if not shown there, in Figure 2.*

This data is subject to the same equations between 2-cells as in a monoidal bicategory (see e.g. *[81]).*

6.4 Premonoidal Kleisli bicategories

We now arrive at our main theorem.

Theorem 2. *For any bistrong strong pseudomonad* (T, s, t) *on a monoidal bicategory* $(\mathscr{B}, \otimes, I)$*, the Kleisli bicategory* $\mathscr{B}_T$ *is premonoidal, with binoidal structure given as in (7).*

Proof. The binoidal structure is as in (7), with s defined as in (5). Then, the proof consists in constructing a compatible pair of a left action and a right action, where "compatible" means that the two associators coincide on 1-cells, and the structural 2-cells $\widetilde{\mathfrak{p}}$ and $\widetilde{\mathfrak{m}}$ coincide. All of this is verified directly, based on the construction of the actions in Theorem 1.

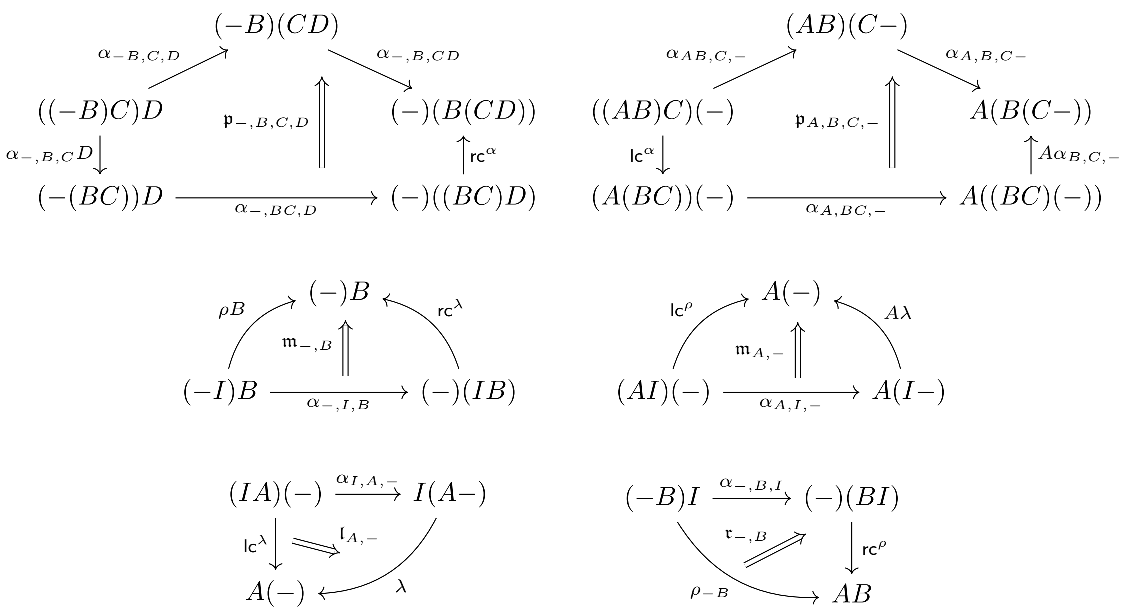


Figure 6: Modification axioms for the structural 2-cells of a premonoidal bicategory, where they differ from those of a monoidal bicategory.

The only remaining difficulty is the pseudonaturality of α in its middle argument, since this is not required for either of the actions. This is where the 2-cell given by the bistrong structure is used. □

By Proposition 3, we immediately get the following.

Corollary 2. *For any strong pseudomonad (T, t) on a symmetric monoidal bicategory $(\mathscr{B}, \otimes, I)$, the Kleisli bicategory $\mathscr{B}_T$ is premonoidal, with binoidal structure given as in (7).*

In constructing this premonoidal structure we make essential use of the following result. For f a morphism in $\mathscr{B}$, write $\tilde{f}$ for the image of f under the pseudofunctor $\mathscr{B} \to \mathscr{B}_T$.

Lemma 3. *For $f \in \mathscr{B}(A, A')$, $\tilde{f} \in \mathscr{B}_T(A, A')$ is canonically central. Moreover, for every 1-cell $g \in \mathscr{B}(B, B')$, $\mathsf{lc}^{\tilde{f}}_{\tilde{g}} = (\mathsf{rc}^{\tilde{g}}_{\tilde{f}})^{-1}$.*

6.4.1 Further examples: graded monads and Para

Another source of examples, which we were not expecting when we began this work, comes from *graded monads* ([79, 59, 43]). A graded monad goes beyond just tracking side-effects to also tracking quantitative information. Formally, a graded monad on $\mathbb{C}$ consists of a monoidal category $(\mathbb{E}, \bullet, I)$ of *grades* and a lax monoidal functor $T : \mathbb{E} \to [\mathbb{C}, \mathbb{C}]$. Thus, one has a functor $T_e : \mathbb{C} \to \mathbb{C}$ for every $e \in \mathbb{E}$, natural in e, together with natural transformations $\phi_{e,e'} : T'_e \circ T_e \Rightarrow T_{e' \bullet e}$ (acting like a monadic multiplication) and $\psi : \mathrm{id} \Rightarrow T_I$ (acting like a monadic unit) satisfying unit and associativity laws. For example, the *graded list monad* L on **Set** has grades given by the strict monoidal category $(\mathbb{N}, \times, 1)$ of natural numbers under multiplication, and $\mathrm{L}_n(X)$ is the set of lists of length at most n. This can be used as a refinement of the usual list monad, with the grade giving an upper bound for the number of branches in a non-deterministic computation (*e.g.* [56]).

Instead of restricting to the symmetric case as we do in Theorem 2, we consider abstract bistrong graded monads. An endofunctor $T : \mathbb{C} \to \mathbb{C}$ equipped with two strengths $t_{A,B} : A \otimes TB \to T(A \otimes B)$

and $s_{A,B} : TA \otimes B \to T(A \otimes B)$ is bistrong if

$$\begin{array}{ccccc} (AT_B)C & \xrightarrow{\alpha} & A(T_B C) & \xrightarrow{As} & AT_{BC} \\ {\scriptstyle tC}\downarrow & & & & \downarrow{\scriptstyle t} \\ T_{AB}C & \xrightarrow[s]{} & T_{(AB)C} & \xrightarrow[T_\alpha]{} & T_{A(BC)} \end{array}$$

commutes. Then, a small adjustment to Katsumata's definition of strong graded monads gives the following.

Definition 18 (*c.f.* [43, Definition 2.5])**.** *A* bistrong graded monad *on a monoidal category* $(\mathbb{C}, \otimes, I)$ *consists of a monoidal category* $(\mathbb{E}, \bullet, I)$ *of grades and a lax monoidal functor* $T : \mathbb{E} \to [\mathbb{C}, \mathbb{C}]_{bistrong}$*, where* $[\mathbb{C}, \mathbb{C}]_{bistrong}$ *is the category of bistrong endofunctors and natural transformations that commute with both strengths (see* e.g. *[57]).*

Thus, a bistrong graded monad is a graded monad further equipped with natural transformations $t^e_{A,B} : A \otimes T_e(B) \to T_e(A \otimes B)$ and $s^e_{A,B} : T_e(A) \otimes B \to T_e(A \otimes B)$ for every grade e, compatible with the graded monad structure and with maps between grades.

Previous work (*e.g.* [56]) has used presheaf enrichment to define Kleisli constructions for graded monads. However, there is also a natural bicategorical construction:

Definition 19. *Let T be a bistrong graded monad on $(\mathbb{C}, \otimes, I)$ with grades $(\mathbb{E}, \bullet, I)$. Define a bicategory $\mathcal{K}_T$ with objects those of $\mathbb{C}$ as follows:*

- *1-cells from A to B are pairs (e, f) consisting of a grade e and a map $f : A \to T_e B$;*
- *2-cells $\gamma : (e, f) \Rightarrow (e', f')$ are maps $\gamma : e \to e'$ in $\mathbb{E}$ such that $T_\gamma(B) \circ f = f'$, with composition as in $\mathbb{E}$;*
- *Composition is defined similarly to Kleisli composition: the composite of $f : A \to T_e B$ and $g : B \to T_{e'} C$ is*
$$A \xrightarrow{f} T_e B \xrightarrow{T_e g} T_e(T_{e'} C) \xrightarrow{\phi} T_{e \bullet e'} C$$
The identity on A is $A \xrightarrow{\psi_A} T_I A$, and the structural isomorphisms are the structural isomorphisms in $\mathbb{E}$.

(One could also present this bicategory as $\mathbf{Para}(\mathbb{C}, \mathbb{E}, *)^{\mathrm{op}}$ where maps are parametrized by the left action of $\mathbb{E}$ on $\mathbb{C}$ with $e * A := T_e(A)$.)

If the graded monad T is bistrong, one obtains strict pseudofunctors $A \rtimes (-), (-) \ltimes B : \mathcal{K}_T \to \mathcal{K}_T$ for every $A, B \in \mathcal{K}_T$, defined similarly to (7). Moreover, just as every map in $\mathbb{C}$ determines a 1-cell in $\mathbf{Para}(\mathbb{C})$, every map $f \in \mathbb{C}(A, A')$ determines a 1-cell with grade I in $\mathcal{K}_T$, as

$$\widetilde{f} := \big(A \xrightarrow{f} A' \xrightarrow{\psi'_A} T_I A'\big).$$

This 1-cell canonically determines a central 1-cell, with $\mathsf{lc}^{\widetilde{f}}$ and $\mathsf{rc}^{\widetilde{f}}$ given by the canonical isomorphism in $\mathbb{C}$. In particular, we get the following.

Lemma 4. *For $f \in \mathbb{C}(A, A')$, $\tilde{f} \in \mathcal{K}_T(A, A')$ is canonically central. Moreover, for every 1-cell $g \in \mathbb{C}(B, B')$, $\mathsf{lc}^{\tilde{f}}_{\tilde{g}} = (\mathsf{rc}^{\tilde{g}}_{\tilde{f}})^{-1}$.*

In a similar fashion, the structural isomorphisms for the monoidal structure on $\mathbb{E}$ define pseudonatural equivalences on $\mathcal{K}_T$ with each component central. These all have grade I, so one can take the structural modifications in $\mathcal{K}_T$ all to be canonical isomorphisms of the form $I^{\otimes i} \xrightarrow{\cong} I^{\otimes j}$ for $i, j \in \mathbb{N}$. Summarising, we have the following.

Proposition 4. *For any bistrong graded monad (T, ϕ, ψ) the bicategory $\mathcal{K}_T$ has a canonical choice of premonoidal structure. Moreover, if $\mathbb{E}$ is strict monoidal, then this structure is strict: each pseudofunctor $A \rtimes (-)$ and $(-) \ltimes B$ is strict, and the structural transformations and structural modifications are all the identity.*

6.5 Aside: bicategories of pure maps

In a premonoidal category $\mathbb{D}$, the central morphisms form a wide subcategory $\mathcal{Z}(\mathbb{D})$ called the *centre*. The definition directly entails that the centre is a monoidal category, and the inclusion $\mathcal{Z}(\mathbb{D}) \hookrightarrow \mathbb{D}$ strictly preserves premonoidal structure. In this section we briefly explore the question of monoidal structure on the centre of a monoidal bicategory. The key difference from the categorical setting is that the bicategorical notion of centrality is *structure* in the form of extra data, not just a *property*: instead of the inclusion $\mathcal{Z}(\mathbb{C}) \hookrightarrow \mathbb{C}$ we now have a forgetful pseudofunctor $\mathcal{Z}(\mathscr{B}) \to \mathscr{B}$. So for the premonoidal structure to induce a monoidal structure, this extra data must also interact well with all the data and equations of a monoidal bicategory.

In this section we will outline a notion of centre for premonoidal bicategories and observe that, because central 1-cells do not satisfy certain equations that always hold in the monoidal case, the centre does straightforwardly acquire a monoidal structure. Then we shall observe that, in our our examples, these equations do in fact hold. By axiomatising this situation, we will recover sufficient conditions for a sub-bicategory of a premonoidal bicategory to be monoidal.

We begin with a natural definition.

Definition 20. *For a premonoidal category $(\mathscr{B}, \rtimes, \ltimes, I)$, denote by $\mathcal{Z}(\mathscr{B})$ the bicategory with the same objects, whose 1-cells and 2-cells are the central 1-cells and central 2-cells in $\mathscr{B}$. Composition of 1-cells and 2-cells is defined using composition in $\mathrm{Hom}(\mathscr{B}, \mathscr{B})$, and the identity on A is Id_A with the identity transformations. The structural 2-cells a, l and r are all inherited from $\mathscr{B}$.*

This definition does satisfy at least some of the expected properties. For example, for any premonoidal bicategory $(\mathscr{B}, \rtimes, \ltimes, I)$ there exist strict pseudofunctors

$$\mathsf{H}_l, \mathsf{H}_r : \mathcal{Z}(\mathscr{B}) \to \mathrm{Hom}(\mathscr{B}, \mathscr{B}) \tag{11}$$

where H_l is given on objects by $\mathsf{H}_l(A) := A \rtimes (-)$, on 1-cells by setting $\mathsf{H}_l(f)$ to be the pseudonatural transformation $(f \ltimes (-), \mathsf{lc}^f)$, and on 2-cells by setting $\mathsf{H}_l(\sigma)$ to be the modification with components $\{\sigma \ltimes B \mid B \in \mathscr{B}\}$; the other pseudofunctor H_r is defined likewise. In a similar vein, we have the next result.

Proposition 5. *Let $(\mathscr{B}, \rtimes, \ltimes, I)$ be a premonoidal bicategory. For every object $A \in \mathscr{B}$ the operations $A \rtimes (-)$ and $(-) \ltimes B$ induce pseudofunctors on $\mathcal{Z}(\mathscr{B})$.*

Proof sketch. The action of $A \rtimes (-)$ and $(-) \ltimes B$ on central 1-cells is given by requiring that the following diagrams define modifications in $\mathrm{Hom}(\mathscr{B}, \mathscr{B})$:

$$\begin{array}{ccc} (XA)(-) & \xrightarrow{\mathsf{lc}^{X \rtimes f}} & (XA')(-) \\ {\scriptstyle \alpha_{X,A,-}}\big\downarrow & \overline{\alpha}_{X,f,-} & \big\downarrow{\scriptstyle \alpha_{X,A',-}} \\ X(A-) & \xrightarrow[X \rtimes \mathsf{lc}^f]{} & X(A'-) \end{array} \qquad\qquad \begin{array}{ccc} (AX)(-) & \xrightarrow{\mathsf{lc}^{f \ltimes X}} & (A'X)(-) \\ {\scriptstyle \alpha_{A,X,-}}\big\downarrow & \overline{\alpha}_{f,X,-} & \big\downarrow{\scriptstyle \alpha_{A',X,-}} \\ A(X-) & \xrightarrow[\mathsf{lc}^f_{X-}]{} & A'(X-) \end{array}$$

Long diagram chases then show that the unitor and compositor for each of $A \rtimes (-)$ and $(-) \ltimes B$ are central, and that these pseudofunctors preserve centrality of 2-cells. ☐

This result suggests that one could hope to restrict the premonoidal structure of $\mathscr{B}$ to a monoidal structure on $\mathcal{Z}(\mathscr{B})$. However, there is a fundamental issue. In a premonoidal category $(\mathbb{C}, \rtimes, \ltimes, I)$ one can show $(f \ltimes B') \circ (A \rtimes g) = (A' \rtimes g) \circ (f \ltimes B)$ using either the fact f is left central or the fact g is right central, and these choices are equivalent. Indeed, the monoidal structure on the centre $\mathcal{Z}(\mathbb{C})$ is defined by setting $(f \otimes g)$ to be either one of these choices. However, the corresponding fact for central 1-cells, namely that $\mathsf{lc}^f_g = (\mathsf{rc}^g_f)^{-1}$, does not hold in general, so one must choose to use either lc or rc to define the compositor of the pseudofunctor $\otimes$. This asymmetry causes a mismatch between $\otimes$ and the axioms of a premonoidal bicategory. For example, the equation for $\mathfrak{r}$ to be a modification on the centre at a pair of 1-cells $f : A \to A'$ and $g : B \to B'$ uses lc^f_ρ—arising from the interchange law for $\otimes$—while the axiom on $\mathfrak{r}$ uses rc^ρ_f (as in Figure 6), so one cannot apply the axiom. Defining the compositor with rc instead runs into the same problem with $\mathfrak{l}$.

On the other hand, this does not seem to be a problem in practice: as our examples above show, in models of effectful languages the 'pure' 1-cells satisfy $\mathsf{lc}^f_g = (\mathsf{rc}^g_f)^{-1}$ and the problem disappears (see Lemma 3 and Lemma 4). In the rest of this section we shall show that, if one axiomatises a version of this equation together with compatibility between the centrality witnesses and the associator, then one does indeed recover a monoidal structure. The idea is to make a coherent assignment of left- and right centrality witness to each 1-cell, and then take 2-cells which satisfy the axioms for centrality with respect to these (but perhaps not all) 1-cells. This can be seen as axiomatising versions of the pseudofunctors of (11) which are compatible with the premonoidal structure. The cost is that, unlike the centre, the resulting structure is not unique: the coherent assignments require a *choice* of centrality witnesses.

Definition 21. *Let $(\mathscr{B}, \rtimes, \ltimes, I)$ be a premonoidal bicategory. A* bicategory of pure maps *is a sub-bicategory $\mathscr{C} \hookrightarrow \mathscr{B}$ with the same objects as $\mathscr{B}$, together with strict pseudofunctors $\mathsf{L}, \mathsf{R} : \mathscr{C} \to \mathrm{Hom}(\mathscr{C}, \mathscr{C})$ such that*

1. *Every structural 1-cell $\alpha_{A,B,C}, \lambda_A, \rho_A$ is a 1-cell in $\mathscr{C}$;*
2. *$\mathsf{L}(A) = A \rtimes (-)$ and $\mathsf{R}(B) = (-) \ltimes B$;*
3. *$\mathsf{L}(A \xrightarrow{f} A')$ is a pseudonatural transformation $(f \ltimes (-), \mathsf{lc}^f)$ with 1-cell components $f \ltimes B : A \otimes B \to A \ \otimes B$, and $\mathsf{R}(B \xrightarrow{g} B')$ is a pseudonatural transformation $((-) \rtimes g, \mathsf{rc}^f)$ with 1-cell components $A \rtimes g : A \otimes B \to A \otimes B'$, and these are compatible with the associator in the sense that the following diagrams are modifications in $\mathrm{Hom}(\mathscr{B}, \mathscr{B})$:*

$$\begin{array}{ccc} (A-)X & \xrightarrow{\mathsf{lc}^f \ltimes X} & (A'-)X \\ {\scriptstyle \alpha_{A,-,X}}\downarrow & \overline{\alpha}_{f,-,X} & \downarrow{\scriptstyle \alpha_{A',-,X}} \\ A(-X) & \xrightarrow[\mathsf{lc}^f_{-X}]{} & A(-X) \end{array} \qquad\qquad \begin{array}{ccc} (X-)A & \xrightarrow{\mathsf{rc}^f_{X-}} & (X-)A' \\ {\scriptstyle \alpha_{X,-,A}}\downarrow & \overline{\alpha}_{X,-,f} & \downarrow{\scriptstyle \alpha_{X,-,A'}} \\ X(-A) & \xrightarrow[X \rtimes \mathsf{rc}^f]{} & X(-A') \end{array}$$

4. *$\mathsf{L}(f \overset{\sigma}{\Longrightarrow} f')$ has components $\sigma \ltimes B$ and $\mathsf{R}(g \overset{\tau}{\Longrightarrow} g')$ has components $A \rtimes \tau$;*
5. *$\overline{\mathsf{L}(f)}_g = \overline{\mathsf{R}(g)}_f$ for all f, g in $\mathscr{C}$.*

We say a 2-cell $\sigma : f \Rightarrow f'$ in $\mathscr{B}$ is central with respect to 1-cells in $\mathscr{C}$ *if $\sigma : f \Rightarrow f'$ in $\mathscr{C}$.*

We can now see the crucial difference between $\mathcal{Z}(\mathscr{B})$ and a bicategory of central maps. Categorically, the inclusion functor $\mathcal{Z}(\mathbb{C}) \hookrightarrow \mathbb{C}$ is faithful, but the corresponding pseudofunctor $\mathcal{Z}(\mathscr{B}) \to \mathscr{B}$ is not injective on 1-cells (locally injective-on-objects): a 1-cell can be central in many ways. This is remedied by bicategories of central maps, because the inclusion $\mathscr{C} \hookrightarrow \mathscr{B}$ is injective on both 1-cells and 2-cells. We thereby recover a situation closer to the categorical setting, and can recover the categorical theory.

The most difficult technical work is contained in the following proposition, which guarantees that the premonoidal structure descends to the bicategory of pure maps. For the first part, one uses the compatibility condition of Definition 21(3) to show that each lc^f_g is a modification $\mathsf{lc}^{(f \ltimes B') \circ (A \rtimes g)} \Rightarrow \mathsf{lc}^{(A' \rtimes g) \circ (f \ltimes B)}$, then uses duality and the equation $\mathsf{lc}^f_g = (\mathsf{rc}^g_f)^{-1}$ to deduce it also satisfies the centrality condition on the right.

Proposition 6. *For any bicategory of pure maps $(\mathscr{C}, \mathsf{L}, \mathsf{R})$ of a premonoidal bicategory $(\mathscr{B}, \rtimes, \ltimes, I)$:*

1. *If $(f, \mathsf{lc}^f, \mathsf{rc}^f)$ and $(g, \mathsf{lc}^g, \mathsf{rc}^f)$ are 1-cells in $\mathscr{C}$ then every 2-cell lc^f_g and rc^f_g is central with respect to 1-cells in $\mathscr{C}$;*
2. *The 2-cells $\overline{\alpha}_{f,B,C}, \overline{\alpha}_{A,g,C}, \overline{\alpha}_{A,B,h}, \overline{\lambda}_f$ and $\overline{\rho}_f$ witnessing pseudonaturality of the structural transformations in $\mathscr{B}$ are central with respect to 1-cells in $\mathscr{C}$;*
3. *The components of the structural modifications $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ and $\mathfrak{r}$ are central with respect to 1-cells in $\mathscr{C}$.*

In the light of the proposition, we can collate the families of pseudofunctors $\{A \rtimes (-)\}_{A\in\mathscr{B}}$ and $\{(-) \ltimes B\}_{B\in\mathscr{B}}$ into a bona fide pseudofunctor of two arguments, and hence define a monoidal structure. We define the tensor $\otimes$ on $\mathscr{C}$ using the theory of *distributive laws between pseudofunctors* of [16]. Suppose given bicategories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ together with families of pseudofunctors $\{A \rtimes (-) : \mathcal{B} \to \mathcal{C}\}_{A\in\mathcal{A}}$ and $\{(-) \ltimes B : \mathcal{A} \to \mathcal{C}\}_{B\in\mathcal{B}}$ that agree on objects in the sense that $A \rtimes B = A \ltimes B$ for all A, B. By [16, Theorem 6.1], to define a pseudofunctor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ extending the mapping on objects, it is sufficient to provide a *distributive law*, consisting of an invertible 2-cell $\delta_{f,g}$: $(f \ltimes B') \circ (A \rtimes g) \Rightarrow (A' \ltimes g) \circ (f \ltimes B)$ for every $f : A \to A'$ and $g : B \to B'$, subject to naturality, unit, and composition axioms. In the case of a premonoidal bicategory, the 2-cells lc^f_g (or rc^f_g) are natural candidates for such a law. Using this, one can construct a monoidal structure in a similar fashion to the categorical setting. Notice that this strategy does not succeed on $\mathcal{Z}(\mathscr{B})$: although we do not have a concrete counterexample, we believe lc^f_g is not generally right central, so this strategy is not even well-typed on $\mathcal{Z}(\mathscr{B})$.

Theorem 3. *For any bicategory of pure maps $(\mathscr{C}, \mathsf{L}, \mathsf{R})$ of a premonoidal bicategory $(\mathscr{B}, \rtimes, \ltimes, I)$:*

1. *Setting $\delta_{f,g} := \mathsf{lc}^f_g$ defines a distributive law on $\mathscr{C}$ in the sense of [16], so the mapping $\otimes$ on objects extends to a pseudofunctor with*

$$(f \otimes g) := \left(A \otimes B \xrightarrow{f \ltimes B} A' \otimes B \xrightarrow{A' \rtimes g} A' \otimes B'\right)$$

   and similarly on 2-cells; the compositor is constructed using the compositors for the binoidal structure and the left centrality witnesses.

2. *The structural transformations and structural modifications for the premonoidal structure on $\mathscr{B}$ define corresponding pseudonatural transformations and modifications on $\mathscr{C}$ with respect to the pseudofunctor $\otimes$.*

Hence, the premonoidal structure of $\mathscr{B}$ restricts to a monoidal structure $(\otimes, I)$ on $\mathscr{C}$, and the pseudofunctors L and R canonically extend to monoidal pseudofunctors.

Remark 3. *This theorem gives a structure reminiscent of a* Freyd category *([74, 52]),* i.e. *a premonoidal category equipped with a chosen monoidal category to act as the centre. However, there is a conceptual difference: here we have started from a notion of purity and constructed a monoidal bicategory, while in a Freyd category the monoidal structure is given as part of the data. We conjecture that these two approaches are closely related, and will explore this in future work.*

7 Conclusion

We have introduced strong pseudomonads and premonoidal bicategories, and validated these definitions with examples and theorems paralleling the categorical setting:

- Certain pseudomonads are canonically strong: for instance, any pseudomonad on $(\mathbf{Cat}, \times)$, any pseudomonad with respect to a cocartesian structure $(0, +)$, or any pseudomonad of the form $(-) \otimes M$ for M a pseudomonoid (Section 3.3);
- A left strength for a pseudomonad is equivalent to a left action on the Kleisli bicategory extending the monoidal structure (Theorem 1);
- A left strong monad on a symmetric monoidal bicategory is also right strong (Proposition 3);
- The Kleisli bicategory for a strong pseudomonad is premonoidal (Theorem 2).

Thus, as outlined in Section 1.3, we can have a degree of confidence that our definition is not too strict to capture examples one would expect, and not too weak to develop bicategorical versions of the categorical theory.

Our contributions can be understood in two ways: as a practical tool towards new semantic models for programming languages, or from the theoretical perspective of categorical algebra and higher category theory.

7.1 Perspectives in semantics

An important inspiration for this work was the theory of *dialogue categories* ([61, 58]), which explains models of linear logic and game semantics in terms of Moggi-style computational effects. A dialogue category is a symmetric monoidal category $\mathbb{C}$ equipped with a negation functor $\neg : \mathbb{C} \longrightarrow \mathbb{C}^{\mathrm{op}}$ such that the double negation functor $\neg\neg : \mathbb{C} \longrightarrow \mathbb{C}$ extends to a strong monad. With our definition of strong pseudomonads, we can incorporate recent bicategorical models (*e.g.* [8, 29, 27, 60, 18]) into this theory. There are also connections to graded monads: see [59, 26].

Future work. A natural step would be to bicategorify the definition of *Freyd categories*, which model effectful languages with an explicit collection of pure *values* ([74, 52]). This would build on Section 6.5, where bicategories of pure maps should provide a leading example. Such a definition should be validated by a appropriate correspondence theorem, in the style of our Theorem 1, lifting the correspondence between actions and Freyd categories (*e.g.* [51, Appendix B]) to the bicategorical setting. In a similar vein, models of higher-order call-by-value languages can be presented as *closed* Freyd categories [75]. These are equivalent to a strong monad on a monoidal category with enough closed structure ([52, 62]). One would therefore expect corresponding results relating closed Freyd bicategories to strong pseudomonads. Further work is also needed to connect the Kleisli bicategory for a graded monad presented here with existing Kleisli constructions such as that in [56].

In the spirit of Moggi's programme ([63, 64]) it would be of interest to extract a 2-dimensional monadic metalanguage as an internal language for our bicategorical models. Existing work already covers the simply-typed λ-calculus [21].

Finally we note that, although higher-categorical definitions can sometimes be intimidating or intractable, it is often possible (and generally preferable) to identify universal properties that make the axioms hold automatically. It would be of use, therefore, to extend the framework of [85] for constructing symmetric monoidal bicategories to the structures introduced here: the similarities in the construction of the strong pseudomonads on $\mathbf{Para}(\mathbb{C})$ (Example 7) and $\mathbf{Span}(\mathbb{C})$ (Corollary 1) suggest this is possible.

7.2 Perspectives in category theory and universal algebra

At a higher level, this work contributes to recent research on Kleisli bicategories and the theory of substitution ([37, 19, 20, 28]). Perhaps the closest work to that presented here is the work of Hyland & Power [38], which introduces a notion of pseudo-commutativity for strict bistrong 2-monads.

This work is also part of a lively line of research studying various kinds of weak monoidal structure in higher dimensions ([4, 5, 24, 28]). From this perspective, the complexities with the 2-cells in the definition of premonoidal bicategories and their centres are perhaps unsurprising, since the interaction between the "funny" tensor product (used to define premonoidal categories [73]), higher-dimensional structure, and interchange laws is subtle, *e.g.* the funny tensor product does not extend to a tensor on the 2-category **Cat** (also see *e.g.* [11, 3, 5]). Our definition of centre also bears some similarities to the Drinfeld centre of a 2-category [12]; the difference being that there one restricts to *objects* A equipped with an isomorphism $A \otimes (-) \cong (-) \otimes A$.

Future work. There are many accounts of strong monads in category theory, for example in terms of enriched structure ([45, 46]). In this paper we have given a direct and explicit definition, but it would be interesting to reconstruct the full theory. Similarly, we conjecture that Hyland & Power's definition of pseudo-commutativity for strict 2-monads will generalise smoothly to our setting, giving examples which would be relevant for semantics (*c.f.* [64, 46]).

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References

[1] J. C. Baez, B. Fong, and B. S. Pollard. A compositional framework for Markov processes. *Journal of Mathematical Physics*, 57(3):033301, 2016.

[2] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77, Berlin, Heidelberg, 1967. Springer Berlin Heidelberg.

[3] J. Bourke and N. Gurski. A cocategorical obstruction to tensor products of Gray-categories. *Theory and Applications of Categories*, 30(11):387–409, 2015. Available at http://www.tac.mta.ca/tac/volumes/30/11/30-11.pdf.

[4] J. Bourke and N. Gurski. The Gray tensor product via factorisation. *Applied Categorical Structures*, 25(4):603–624, 2016.

[5] J. Bourke and G. Lobbia. A skew approach to enrichment for Gray-categories, 2022.

[6] M. Capucci, B. Gavranović, J. Hedges, and E. Fjeldgren Rischel. Towards foundations of categorical cybernetics. *Electronic Proceedings in Theoretical Computer Science*, 372:235–248, 2022.

[7] A. Carboni, S. Lack, and R.F.C. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, February 1993.

[8] S. Castellan, P. Clairambault, S. Rideau, and G. Winskel. Games and Strategies as Event Structures. *Logical Methods in Computer Science*, Volume 13, Issue 3, September 2017.

[9] E. Cheng and N. Gurski. The periodic table of n-categories II: degenerate tricategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 52(2), 2011. Available at http://www.numdam.org/item/CTGDC_2011__52_2_82_0/.

[10] E. Cheng, M. Hyland, and J. Power. Pseudo-distributive laws. *Electronic Notes in Theoretical Computer Science*, 83:227–245, 2003.

[11] S. Crans. A tensor product for Gray-categories. *Theory and Applications of Categories*, 5(2):12–69, 1999. Available at http://www.tac.mta.ca/tac/volumes/1999/n2/n2.pdf.

[12] S. E. Crans. Generalized centers of braided and sylleptic monoidal 2-categories. *Advances in Mathematics*, 136(2):183–223, 1998.

[13] I.J. Le Creurer, F. Marmolejo, and E.M. Vitale. Beck's theorem for pseudo-monads. *Journal of Pure and Applied Algebra*, 173(3):293–313, 2002.

[14] G. S. H. Cruttwell, B. Gavranović, N. Ghani, P. Wilson, and F. Zanasi. Categorical foundations of gradient-based learning. In *Programming Languages and Systems*, pages 1–28. Springer International Publishing, 2022.

[15] B. Day and R. Street. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129(1):99–157, 1997.

[16] P. F. Faul, G. Manuell, and J. Siqueira. 2-dimensional bifunctor theorems and distributive laws. *Theory and Applications of Categories*, 37(34):1149–1175, 2021.

[17] J. L. Fiadeiro and V. Schmitt. Structured co-spans: An algebra of interaction protocols. In *Algebra and Coalgebra in Computer Science*, pages 194–208. Springer Berlin Heidelberg, 2007.

[18] M. Fiore, Z. Galal, and H. Paquet. A combinatorial approach to higher-order structure for polynomial functors. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

[19] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. The cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society*, 77(1):203–220, 2007.

[20] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. *Selecta Mathematica*, 24(3):2791–2830, Jul 2018.

[21] M. Fiore and P. Saville. A type theory for cartesian closed bicategories. In *Proceedings of the Thirty-Fourth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2019.

[22] M. Fiore and P. Saville. Coherence for bicategorical cartesian closed structure. *Mathematical Structures in Computer Science*, 31(7):822–849, August 2021.

[23] B. Fong, D. Spivak, and R. Tuyeras. Backprop as functor: A compositional perspective on supervised learning. In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, jun 2019.

[24] S. Forest and S. Mimram. Free precategories as presheaf categories, 2022.

[25] C. Führmann. *The structure of call-by-value*. PhD thesis, University of Edinburgh, 2000.

[26] S. Fujii, S. Katsumata, and P-A. Melliès. Towards a formal theory of graded monads. In *Lecture Notes in Computer Science*, pages 513–530. Springer Berlin Heidelberg, 2016.

[27] Z. Galal. A profunctorial Scott semantics. In *Proceedings of the 5th International Conference on Formal Structures for Computation and Deduction (FSCD 2020)*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[28] N. Gambino, R. Garner, and C. Vasilakopoulou. Monoidal kleisli bicategories and the arithmetic product of coloured symmetric sequences, 2022.

[29] N. Gambino and A. Joyal. On operads, bimodules and analytic functors. *Memoirs of the American Mathematical Society*, 249(1184):153–192, 2017.

[30] R. Garner. Polycategories via pseudo-distributive laws. *Advances in Mathematics*, 218(3):781–827, 2008.

[31] F. R. Genovese, J. Herold, F. Loregian, and D. Palombi. A categorical semantics for hierarchical Petri nets. *Electronic Proceedings in Theoretical Computer Science*, 350:51–68, 2021.

[32] R. Gordon, A. J. Power, and R. Street. *Coherence for tricategories*. Memoirs of the American Mathematical Society, 1995.

[33] N. Gurski. *Coherence in Three-Dimensional Category Theory*. Cambridge University Press, 2013.

[34] N. Gurski and A. Osorno. Infinite loop spaces, and coherence for symmetric monoidal bicategories. *Advances in Mathematics*, 246:1 – 32, 2013.

[35] C. Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164–225, 2000.

[36] B.P. Hilken. Towards a proof theory of rewriting: the simply typed 2λ-calculus. *Theoretical Computer Science*, 170(1):407–444, 1996.

[37] J.M.E. Hyland. Elements of a theory of algebraic theories. *Theoretical Computer Science*, 546:132–144, 2014.

[38] M. Hyland and J. Power. Pseudo-commutative monads and pseudo-closed 2-categories. *Journal of Pure and Applied Algebra*, 175(1-3):141–185, 2002.

[39] G. Janelidze and G. M. Kelly. A note on actions of a monoidal category. *Theory and Applications of Categories*, 9(4):61–91, 2001. Available at tac.mta.ca/tac/volumes/9/n4/n4.pdf.

[40] A. Jeffrey. Premonoidal categories and a graphical view of programs. *Preprint, Dec*, pages 80688–7, 1997.

[41] N. Johnson and D. Yau. *2-Dimensional Categories*. Oxford University Press, 2021.

[42] A. Joyal and R. Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 11 1993.

[43] S. Katsumata. Parametric effect monads and semantics of effect systems. In *Proceedings of the 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. ACM, 2014.

[44] A. Kerinec, G. Manzonetto, and F. Olimpieri. Why are proofs relevant in proof-relevant models? *Proceedings of the ACM on Programming Languages*, 7(POPL):218–248, 2023.

[45] A. Kock. Monads on symmetric monoidal closed categories. *Archiv der Mathematik*, 21(1):1–10, 1970.

[46] A. Kock. Strong functors and monoidal monads. *Archiv der Mathematik*, 23(1):113–120, 1972.

[47] S. Lack. A coherent approach to pseudomonads. *Advances in Mathematics*, 152(2):179–202, 2000.

[48] S. Lack. Icons. *Applied Categorical Structures*, 18(3):289–307, apr 2008.

[49] S. Lack, R. F. C. Walters, and R. J. Wood. Bicategories of spans as cartesian bicategories. *Theory and Applications of Categories*, 24(1), 2010.

[50] T. Leinster. Basic bicategories. Available at https://arxiv.org/abs/math/9810017, May 1998.

[51] P. B. Levy. *Call-By-Push-Value: A Functional/Imperative Synthesis*. Springer Netherlands, 2003.

[52] P. B. Levy, J. Power, and H. Thielecke. Modelling environments in call-by-value programming languages. *Information and Computation*, 185(2):182–210, September 2003.

[53] S. Mac Lane and R. Paré. Coherence for bicategories and indexed categories. *Journal of Pure and Applied Algebra*, 37:59 – 80, 1985.

[54] F. Marmolejo. Doctrines whose structure forms a fully faithful adjoint string. *Theory and Applications of Categories*, 3(2):23–24, 1997.

[55] F. Marmolejo. Distributive laws for pseudomonads. *Theory and Applications of Categories*, 5(5), 1999. Available at http://www.tac.mta.ca/tac/volumes/1999/n5/n5.pdf.

[56] D. McDermott and T. Uustalu. Flexibly graded monads and graded algebras. In *Lecture Notes in Computer Science*, pages 102–128. Springer International Publishing, 2022.

[57] D. McDermott and T. Uustalu. What makes a strong monad? *Electronic Proceedings in Theoretical Computer Science*, 360:113–133, jun 2022.

[58] P.-A. Melliès. Game semantics in string diagrams. In *2012 27th Annual IEEE Symposium on Logic in Computer Science*, pages 481–490. IEEE, 2012.

[59] P.-A. Melliès. Parametric monads and enriched adjunctions. Available at https://www.irif.fr/~mellies/tensorial-logic/8-parametric-monads-and-enriched-adjunctions.pdf., 2012.

[60] P.-A. Melliès. Asynchronous template games and the Gray tensor product of 2-categories. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, jun 2021.

[61] P.-A. Melliès and N. Tabareau. Resource modalities in game semantics. *CoRR*, abs/0705.0462, 2007.

[62] R. Møgelberg and S. Staton. Linear usage of state. *Logical Methods in Computer Science*, 10(1), 2014.

[63] E. Moggi. Computational lambda-calculus and monads. In *Proceedings, Fourth Annual Symposium on Logic in Computer Science*. IEEE Comput. Soc. Press, 1989.

[64] E. Moggi. Notions of computation and monads. *Information and Computation*, 93(1):55–92, 1991.

[65] F. Olimpieri. Intersection type distributors. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2021.

[66] H. Paquet. *Probabilistic concurrent game semantics*. PhD thesis, University of Cambridge, 2019.

[67] B Pareigis. Non-additive ring and module theory II. C-categories, C-functors and C-morphisms. *Publ. Math. Debrecen*, 24(3–4):351–361, 1977.

[68] P. Perrone and W. Tholen. Kan extensions are partial colimits. *Applied Categorical Structures*, 30(4):685–753, 2022.

[69] A. J. Power. Coherence for bicategories with finite bilimits I. In J. W. Gray and A. Scedrov, editors, *Categories in Computer Science and Logic: Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference Held June 14–20, 1987 with Support from the National Science Foundation*, volume 92, pages 341–349. American Mathematical Society, 1989.

[70] J. Power. Why tricategories? *Inf. Comput.*, 120(2):251–262, August 1995.

[71] J. Power. 2-categories. *BRICS Notes Series*, 1998.

[72] J. Power. Premonoidal categories as categories with algebraic structure. *Theoretical Computer Science*, 278(1-2):303–321, 2002.

[73] J. Power and E. Robinson. Premonoidal categories and notions of computation. *Mathematical Structures in Computer Science*, 7(5):453–468, October 1997.

[74] J. Power and H. Thielecke. Environments, continuation semantics and indexed categories. In *Lecture Notes in Computer Science*, pages 391–414. Springer Berlin Heidelberg, 1997.

[75] J. Power and H. Thielecke. Closed Freyd- and κ-categories. In *Automata, Languages and Programming*, pages 625–634. Springer Berlin Heidelberg, 1999.

[76] M. Román. Promonads and string diagrams for effectful categories. *CoRR*, abs/2205.07664, 2022.

[77] H. Paquet S. Dash, Y. Kaddar and S. Staton. Affine monads and lazy structures for bayesian programming. *50th ACM SIGPLAN Symposium on Principles of Programming Languages (POPL 2023)*, 2023 (to appear).

[78] C. J. Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, University of California, 2009. Available at arxiv.org/pdf/1112.1000.pdf.

[79] A. L. Smirnov. Graded monads and rings of polynomials. *Journal of Mathematical Sciences*, 151(3):3032–3051, 2008.

[80] S. Staton. Commutative semantics for probabilistic programming. In *Programming Languages and Systems*, pages 855–879. Springer Berlin Heidelberg, 2017.

[81] M. Stay. Compact closed bicategories. *Theories and Applications of Categories*, 31(26):755–798, 2016.

[82] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149 –168, 1972.

[83] H. Thielecke. Continuation semantics and self-adjointness. *Electronic Notes in Theoretical Computer Science*, 6:348–364, 1997.

[84] T. Tsukada, K. Asada, and C.-H. L. Ong. Species, profunctors and Taylor expansion weighted by SMCC. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. ACM, jul 2018.

[85] L. Wester Hansen and M. Shulman. Constructing symmetric monoidal bicategories functorially. *arXiv preprint arXiv:1910.09240*, 2019.